

Global Solutions to a New Keynesian Model*

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Abstract

We construct a continuum of stationary equilibria in a continuous-time New Keynesian model with a conventional Taylor rule. Our earlier paper ([Lee and Dordal i Carreras, 2026](#)) establishes multiple equilibria with a linearized New Keynesian Phillips curve; here, we show that they persist when the supply block is fully non-linear, and admit a closed-form solution. The model retains precautionary saving induced by consumption risk, while firms face quadratic Rotemberg costs that make the Phillips curve non-linear in inflation and the output gap. Every equilibrium in the family features underproduction, deflation, and positive excess volatility. Although output fluctuates persistently within a bounded range, it features a stationary distribution, with the shape of the inflation function determining how output and inflation behave over time. Finally, the constructed equilibria are sustained even when an effective lower bound on the policy rate binds at low output gaps.

Keywords: New Keynesian model, Taylor rules, self-fulfilling volatility, global equilibrium, stationary equilibrium.

JEL codes: E31, E32, E52.

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1 Introduction

A sufficiently strong response of the nominal interest rate to inflation is commonly associated with local determinacy in the New Keynesian models (Woodford, 2003; Galí, 2015). However, local determinacy need not imply uniqueness among globally bounded equilibria. In our earlier paper (Lee and Dordal i Carreras, 2026), households that anticipate greater consumption volatility save more for precautionary reasons, reducing aggregate demand. Under nominal rigidities, the resulting decline in real activity validates the anticipated volatility. The model admits bounded equilibria driven by self-fulfilling volatility, with persistent underproduction, including a stationary class.

This paper retains the economic environment, monetary policy following Taylor rules, and demand block of Lee and Dordal i Carreras (2026), but replaces its log-linearized supply block with an exact non-linear New Keynesian Phillips curve. Firms incur quadratic Rotemberg price-adjustment costs (Rotemberg, 1981), which are rebated lump-sum to households. Despite the model’s non-linearities, we show that it continues to admit a continuum of stationary equilibria. In every constructed equilibrium, the output gap and inflation are strictly negative, while excess output-growth volatility is strictly positive. The output gap follows a positive recurrent diffusion on a bounded interval, and the shape of the inflation function determines the stationary distribution. Sample paths can approach arbitrarily closely to both endpoints but never reach either from the interior. Despite these persistent fluctuations, the invariant distribution implies well-defined long-run averages of the output gap, inflation, and excess volatility.

Surprisingly, the multiplicity persists when nominal interest rates are subject to a constant effective lower bound that binds occasionally. In each constrained equilibrium, the economy spends positive stationary probabilities in both the constrained and unconstrained regimes. This is similar to the alternative zero-lower-bound equilibria driven by self-fulfilling volatility in Dordal i Carreras and Lee (2026).¹

1.1 Related literature

A sufficiently aggressive response of the policy rate to inflation or the output gap ensures local determinacy in a canonical New Keynesian model (Clarida, Galí, and Gertler, 1999; Woodford, 2001). As Cochrane (2011) argues, however, uniqueness hinges on ruling out explosive paths

¹In Dordal i Carreras and Lee (2026), certain stabilization at a future date removes current excess volatility and selects a unique equilibrium. In the stationary economy constructed here, every future date repeats the same policy: the nominal rate follows the Taylor rule or binds at the effective lower bound, according to the current state. Under that recurrence, the stationary equilibria persist, and excess volatility remains positive.

through a boundedness restriction.² Beyond the local equilibrium, boundedness alone need not ensure uniqueness.

Our earlier paper ([Lee and Dordal i Carreras, 2026](#)) studies this issue in continuous time under complete information. With a log-linearized Phillips curve, households that anticipate greater consumption volatility save more, reducing demand and validating their expectations. The resulting bounded equilibria include the stationary underproduction class studied here. Two extensions restore uniqueness. A small friction in social memory, in the sense of [Angeletos and Lian \(2023\)](#), eliminates the self-fulfilling paths, while level- k reasoning, as in [Farhi and Werning \(2019\)](#), weakens the intertemporal coordination that sustains them. In a separate overlapping-generations extension, non-Ricardian fiscal policy selects the same equilibrium. Still, the non-linear New Keynesian Phillips curve is used only for numerical solutions there, and the effective lower bound constraint is not imposed. [Khorrami and Mendo \(2026\)](#) similarly show how active fiscal policy can eliminate multiple equilibria.³

A separate literature connects multiplicity to idiosyncratic income risk. In an analytical heterogeneous-agent model, [Acharya and Dogra \(2020\)](#) show that countercyclical risk can generate indeterminacy even when the nominal interest rate satisfies the Taylor principle, whereas procyclical risk can make a nominal peg determinate. [Acharya and Benhabib \(2026\)](#) establish a corresponding global result in a Heterogeneous-Agent New Keynesian (HANK) economy. When risk is even mildly countercyclical, the natural rate becomes endogenous and moves with output, allowing multiple bounded equilibria to survive arbitrarily strong responses of the policy rate to inflation. An active monetary policy rule eliminates multiplicity if it stabilizes these natural-rate movements; a passive monetary policy combined with active fiscal policy can do so too. Our focus is different in that households are representative, and the natural rate is exogenously given.⁴ We construct a family of stationary equilibria based on self-fulfilling aggregate volatility, and in particular, show a fully non-linear New Keynesian Phillips curve supports a continuum of stationary equilibria with stochastic aggregate volatility.

The persistence of the stationary equilibrium family under an effective lower bound relates to the liquidity-trap equilibria of [Benhabib, Schmitt-Grohé, and Uribe \(2002\)](#). With a zero lower bound, a Taylor rule can admit self-fulfilling paths of declining inflation along which monetary policy fails to restore the target. In Section 3.3, the lower bound constrains monetary policy only over part of the state space, leaving the Taylor rule region operative elsewhere. The continuum of stationary equilibria nevertheless persists.

²Local determinacy in linear rational-expectations models is commonly established using the criterion of [Blanchard and Kahn \(1980\)](#).

³Their equilibrium retains the precautionary term in the IS equation and mainly uses a linearized supply block. The fully non-linear pricing block appears only for studying deterministic dynamics.

⁴Still, the risk-adjusted natural rate responds to endogenous volatility ([Lee and Dordal i Carreras, 2026](#)).

The paper is organized as follows. Section 2 sets out the economy. Section 2.7 derives the exact New Keynesian Phillips curve and IS equation. Section 3 constructs the stationary family and its invariant density. In particular, Section 3.2 presents bounds, cyclical comovement, and long-run averages. Section 3.3 shows that an occasionally binding effective lower bound preserves equilibrium construction. Section 4 concludes.

2 The model

The economy consists of a unit measure of identical households and a fixed unit continuum of monopolistically competitive firms, indexed by $h, j \in [0, 1]$, respectively. Time is continuous. The aggregate price level, nominal wage, and nominal short rate are denoted by p_t , w_t , and i_t . Aggregate processes are adapted to a filtration $(\mathcal{F}_t)$ satisfying the usual conditions, and Z_t is a standard Brownian motion on this filtration; $\mathbb{E}_t$ denotes conditional expectation. Aggregate inflation is defined by $dp_t/p_t = \pi_t dt$, with $p_0 > 0$, and the real short rate is $r_t = i_t - \pi_t$. Unless stated otherwise, functions of the state are evaluated at the current state. The nominal bond is the only traded security. As usual, firms are owned by households, and their profits are rebated back to households in a lump-sum way.

2.1 Households

Household h chooses consumption $C_t^h > 0$, labor supply $L_t^h \geq 0$ and nominal bond B_t^h . Let $C_t^{h,j}$ be its consumption of firm j 's good, whose nominal price is p_t^j . C_t^h is then given by:

$$C_t^h = \left[\int_0^1 (C_t^{h,j})^{(\varepsilon-1)/\varepsilon} dj \right]^{\varepsilon/(\varepsilon-1)}, \quad \varepsilon > 1.$$

The elasticity of substitution across goods is ε (Dixit and Stiglitz, 1975).⁵

Let Γ_t^h denote maximal expected continuation utility at date t , given inherited wealth and aggregate information, and let s index future dates:

$$\Gamma_t^h = \sup \mathbb{E}_t \int_t^\infty e^{-\rho(s-t)} \left[\log C_s^h - \frac{(L_s^h)^{1+1/\eta}}{1+1/\eta} \right] ds, \quad \rho > 0. \quad (2.1)$$

The subjective discount rate is $\rho > 0$, and the Frisch elasticity of labor supply is finite and positive, $0 < \eta < \infty$. Write $m = 1 + 1/\eta > 1$ and $U(C, L) = \log C - L^m/m$ for period utility. The preference specification is the same as in Lee and Dordal i Carreras (2026).

⁵Expenditure minimization yields nominal consumption expenditure $p_t C_t^h$. Aggregate consumption is $C_t = \int_0^1 C_t^h dh$.

The nominal transfer D_t collects rebates in firm profits and price adjustment costs, defined below. Households take aggregate wages, prices, and transfers as given, and bonds are in zero net supply. Then, the nominal budget constraint is

$$dB_t^h = (i_t B_t^h - p_t C_t^h + w_t L_t^h + D_t)dt, \quad B_0^h = 0. \quad (2.2)$$

Given aggregate consumption C_t , we define nominal and real discounted marginal utility by $\xi_t^N = e^{-\rho t}/(p_t C_t)$ and $\xi_t = p_t \xi_t^N$. Household plans must satisfy the transversality condition (Dybvig and Huang, 1988; Acemoglu, 2009):

$$\liminf_{T \rightarrow \infty} \mathbb{E}_0[\xi_T^N B_T^h] \geq 0. \quad (2.3)$$

2.2 Production and technology

Let Y_t^j and L_t^j be firm-level output and labor. Aggregate output Y_t uses the same aggregator as consumption, and aggregate labor is $L_t = \int_0^1 L_t^j dj$. Every firm has the technology $Y_t^j = A_t L_t^j$, where common labor productivity follows

$$\frac{dA_t}{A_t} = g dt + \sigma dZ_t, \quad A_0 > 0, \quad \sigma > 0. \quad (2.4)$$

The initial levels A_0 and p_0 are positive constants. Both the drift g and volatility σ are finite, and $\sigma > 0$ is the instantaneous volatility of productivity growth.

2.3 Demand and price setting

The expenditure minimization gives the price index and household good demands. Aggregating those demands and clearing goods markets yields⁶

$$\begin{aligned} p_t &= \left[\int_0^1 (p_t^j)^{1-\varepsilon} dj \right]^{1/(1-\varepsilon)}, \\ C_t^{h,j} &= \left(\frac{p_t^j}{p_t} \right)^{-\varepsilon} C_t^h, \quad Y_t^j = \left(\frac{p_t^j}{p_t} \right)^{-\varepsilon} Y_t. \end{aligned} \quad (2.5)$$

Firm j chooses its own inflation rate π_t^j , with $dp_t^j/p_t^j = \pi_t^j dt$, taking aggregate variables as given. Firms pay quadratic price-adjustment costs à la Rotemberg (1981), where $\tau > 0$ is the

⁶Appendix A.2 derives these demands.

adjustment-cost coefficient. The nominal adjustment cost Θ_t^j and net nominal profit Ψ_t^j are

$$\begin{aligned}\Theta_t^j &= \frac{\tau}{2}(\pi_t^j)^2 p_t^j Y_t^j, \\ \Psi_t^j &= p_t^j Y_t^j - w_t L_t^j - \Theta_t^j, \quad \tau > 0.\end{aligned}\tag{2.6}$$

From its initial price p_t^j , the firm maximizes expected discounted real profit,

$$\mathbb{E}_t \int_t^\infty \frac{\xi_s}{\xi_t} \frac{\Psi_s^j}{p_s} ds,\tag{2.7}$$

where the weight ξ_s/ξ_t is the real stochastic discount factor ([Galí, 2015](#)).

2.4 Aggregation and the natural allocation

Aggregate profits and price-adjustment costs are rebated lump-sum. Consequently,

$$D_t = \int_0^1 \Psi_t^j dj + \int_0^1 \Theta_t^j dj, \quad w_t L_t + D_t = p_t Y_t.\tag{2.8}$$

There is no investment or government spending. Goods-market clearing requires $\int_0^1 C_t^{h,j} dh = Y_t^j$ for each firm. Market clearing gives $C_t^h = C_t = Y_t = A_t L_t$ and $L_t^h = L_t = \int_0^1 L_t^j dj$. Real marginal cost is $\text{mc}_t = w_t/(p_t A_t)$, and $\alpha = (\varepsilon - 1)/\varepsilon$ is the inverse of the gross markup. The symmetric household labor condition gives $\text{mc}_t = L_t^m$.

Under flexible prices, [Appendix A.2](#) shows that individual static pricing and then symmetry give $\text{mc}^n = \alpha$, so natural labor and output are $L^n = \alpha^{1/m}$ and $Y_t^n = A_t L^n$. The associated natural rate of interest is given by $r^n = \rho + g - \sigma^2$.

The output gap We define the output gap $\tilde{Y}_t$ and gross output ratio q_t are

$$\tilde{Y}_t = \frac{Y_t - Y_t^n}{Y_t^n}, \quad q_t = 1 + \tilde{Y}_t = \frac{Y_t}{Y_t^n},\tag{2.9}$$

leading to

$$\text{mc}_t = \alpha q_t^m, \quad \frac{D_t}{p_t C_t} = 1 - \alpha q_t^m.\tag{2.10}$$

The ratio q_t is the state coordinate used throughout the paper. The Taylor rule is written in terms of the actual output gap $\tilde{Y}_t = q_t - 1$.

2.5 Monetary policy

Monetary policy follows a usual Taylor rule:

$$i_t = r^n + \phi_y \tilde{Y}_t + \phi_\pi \pi_t, \quad \phi_y > 0, \quad \phi_\pi \geq 1. \quad (2.11)$$

The coefficients ϕ_y and ϕ_π are the responses of the nominal rate to the output gap and to inflation (Taylor, 1993).

Except in Section 3.3, in which an effective lower bound constrains the policy prescription, the nominal interest rate follows (2.11) at every state.

2.6 Equilibrium

Definition 1 (Symmetric equilibrium). A symmetric equilibrium consists of household plans $(C_t^h, (C_t^{h,j})_j, L_t^h, B_t^h)_h$, firm plans $(p_t^j, \pi_t^j, Y_t^j, L_t^j)_j$ and aggregate processes $(A_t, p_t, \pi_t, w_t, i_t, D_t, C_t, Y_t, L_t)$. All plans are adapted. Given the aggregate processes and individual states, at every decision date, each household maximizes (2.1) subject to budget (2.2) and the transversality condition (2.3); each firm maximizes its discounted real profit (2.7).

Prices and allocations are symmetric, with $C_t^h = C_t = Y_t = A_t L_t$, $L_t^h = L_t$ and $B_t^h = 0$. Goods, labor, and bond markets clear. Thus, transfers satisfy (2.8), monetary policy satisfies (2.11) and productivity satisfies (2.4). Initial conditions are $A_0, p_0 > 0$, $p_0^j = p_0$ and $B_0^h = 0$.

Let the dynamics of aggregate consumption C_t be given by

$$\frac{dC_t}{C_t} = \mu_t^C dt + \sigma_t^C dZ_t,$$

where μ_t^C is its proportional drift and σ_t^C its proportional diffusion loading. Excess output-growth volatility is $\sigma_t^s = \sigma_t^C - \sigma$ as in Lee and Dordal i Carreras (2026).

Now we define stationary equilibrium.

Definition 2 (Stationary equilibrium). A stationary equilibrium is an equilibrium in Definition 1 for which $(\tilde{Y}_t, \pi_t, \sigma_t^s)$ is jointly strictly stationary: its finite-dimensional distributions are unchanged by any common non-negative shift of the observation dates.

2.7 The New Keynesian Phillips curve and IS equation

The symmetric equilibrium of Definition 1 allows exact New Keynesian Phillips curve and IS equation. Define $\kappa = (\varepsilon - 1)/\tau$. The following proposition presents the optimality conditions for households and firms and derives both equations.

Proposition 1 (Exact optimality conditions). *Household optimality requires*

$$\frac{w_t}{p_t C_t^h} = (L_t^h)^{1/\eta}, \quad (2.12)$$

$$\mathbb{E}_t \left[\frac{d\xi_t^{N,h}}{\xi_t^{N,h}} \right] = -i_t dt, \quad (2.13)$$

where $\xi_t^{N,h} = e^{-\rho t} / (p_t C_t^h)$. For individual consumption-growth loading $\sigma_t^{C,h}$, the Euler equation (2.13) is equivalently

$$\frac{dC_t^h}{C_t^h} = \left[r_t - \rho + (\sigma_t^{C,h})^2 \right] dt + \sigma_t^{C,h} dZ_t. \quad (2.14)$$

After individual differentiation and then symmetry, inflation and the proportional gap obey

$$\mathbb{E}_t[d\pi_t] = \left\{ \rho\pi_t - \kappa \left[(1 + \tilde{Y}_t)^{1+1/\eta} - 1 \right] - \frac{\varepsilon - 1}{2} \pi_t^2 \right\} dt, \quad (2.15)$$

$$d\tilde{Y}_t = (1 + \tilde{Y}_t) \left[i_t - \pi_t - r^n + \sigma\sigma_t^s + (\sigma_t^s)^2 \right] dt + (1 + \tilde{Y}_t) \sigma_t^s dZ_t. \quad (2.16)$$

Proof. Appendix A.1 derives the individual household conditions and the individual pricing equation. Once symmetry is imposed, (2.16) and (2.15) follow from those optimality conditions. $\square$

The variance term in (2.14) is precautionary saving. The two volatility terms in the IS equation also pick up the covariance between consumption and natural output. Equation (2.16) is equivalent to the IS equation in our earlier paper (Lee and Dordal i Carreras, 2026).⁷ In the Phillips curve, marginal cost is non-linear in labor, and the squared-inflation term records the effect of adjustment costs. Linearizing (2.15) at the natural allocation gives

$$\mathbb{E}_t[d\pi_t] = (\rho\pi_t - m\kappa\tilde{Y}_t) dt,$$

with local slopes $\kappa^\pi = \rho$ and $\kappa^y = m\kappa$. Under this linearized Phillips curve, our earlier paper establishes multiple equilibria. The family constructed below satisfies the non-linear equation (2.15) itself.

Equations (2.11), (2.15) and (2.16) are the aggregate 3-equations system. The next section constructs a family of stationary solutions and their invariant distribution.

⁷The output gap $\tilde{Y}_t$ is defined as $\log(Y_t/Y_t^n)$ in Lee and Dordal i Carreras (2026), leading to a slightly different form of the IS equation.

3 A family of stationary equilibria

Our equilibrium construction indexes inflation by two parameters, b and ν , each between zero and one. They label members of the stationary equilibrium family, with household preferences (2.1) and the Taylor rule (2.11) held fixed. The level b scales the rise of inflation from its lower limit toward zero, and the power ν governs the curvature of that rise.

Now we present a family of equilibria based on the two parameters (b, ν) . The family uses four constants. Define the auxiliary rate ω , the lower inflation limit $\underline{\pi}$, the amplitude β , and the limiting real-rate shortfall Φ by

$$\begin{aligned}\omega &= \sqrt{\rho^2 + 2(\varepsilon - 1)\kappa}, \quad \underline{\pi} = \frac{\rho - \omega}{\varepsilon - 1} < 0, \\ \beta &= -b\underline{\pi} > 0, \quad \Phi = \phi_y - (\phi_\pi - 1)\underline{\pi}.\end{aligned}\tag{3.1}$$

We impose the following sufficient restrictions for this construction:

$$\begin{aligned}0 < b < 1, \quad 0 < \nu < 1, \quad \nu^2\Phi < \omega, \\ \beta\nu(\phi_\pi - 1) < \rho - \frac{\varepsilon - 1}{2}(1 - b)\underline{\pi}.\end{aligned}\tag{3.2}$$

For any fixed model parameters and for each fixed $b \in (0, 1)$, every sufficiently small positive ν satisfies the inequalities in (3.2). The role of each inequality appears below.

For a positive output ratio q , define first the inflation function $\pi(q)$, real-rate deviation $H(q) \equiv i_t - \pi_t - r^n$ and auxiliary function $F_\eta(q)$ by

$$\pi(q) = \underline{\pi} + \beta q^\nu, \quad H(q) = \phi_y(q - 1) + (\phi_\pi - 1)\pi(q),\tag{3.3}$$

$$F_\eta(q) = \frac{\omega}{\nu} - \frac{\kappa}{\beta\nu}q^{1+1/\eta-\nu} - \frac{(\varepsilon - 1)\beta}{2\nu}q^\nu - H(q).\tag{3.4}$$

The lower inflation limit $\underline{\pi}$ is the negative root of $\rho\underline{\pi} + \kappa - (\varepsilon - 1)\underline{\pi}^2/2 = 0$.⁸ Under (3.2), Lemma 1 in Appendix A.3 shows that F_η is strictly decreasing and has a unique zero $u_\eta \in (0, 1)$. The last inequality in (3.2) is exactly that root condition, so the state interval ends at u_η , which is strictly below natural output ($q = 1$). In other words, the state space is given by $q \in (0, u_\eta)$.⁹

The inequality $\nu^2\Phi < \omega$ makes the drift of the log output ratio $\log q$ positive near zero, which is part of the boundary argument in Appendix A.4. Overall, F_η is positive on $(0, u_\eta)$,

⁸Choosing the negative root here selects the deflationary family of equilibria.

⁹In Section 3.1 and Appendix A.4, we show that boundaries 0 and u_η are not reached in finite time from an interior point.

where excess volatility is given by

$$\sigma_\eta^s(q) = \frac{\sqrt{\sigma^2 + 2(1 + \nu)F_\eta(q)} - \sigma}{1 + \nu}. \quad (3.5)$$

which is strictly decreasing on $q \in (0, u_\eta)$ as $F_\eta(q)$ is strictly decreasing as well.¹⁰

On this interval $q \in (0, u_\eta)$, define the state drift μ_η , diffusion loading ζ_η and candidate state process by

$$\begin{aligned} \mu_\eta(q) &= q[H(q) + \sigma\sigma_\eta^s(q) + (\sigma_\eta^s(q))^2], \quad \zeta_\eta(q) = q\sigma_\eta^s(q), \\ dq_t &= \mu_\eta(q_t)dt + \zeta_\eta(q_t)dZ_t, \quad \pi_t = \pi(q_t), \quad \sigma_t^s = \sigma_\eta^s(q_t). \end{aligned} \quad (3.6)$$

which is equivalent to the exact IS equation (2.16), given $r_t - r^n = H(q_t)$, which itself is from the Taylor rule (2.11).

For a twice continuously differentiable test function ψ , the generator is

$$\mathcal{L}_\eta\psi(q) = \mu_\eta(q)\psi'(q) + \zeta_\eta(q)^2\psi''(q)/2.$$

Below, $\mathbb{E}_q$ denotes expectation from a fixed initial value $q_0 = q$.

The full candidate allocation is $L_t^h = L_t^j = L_t = L^n q_t$, $C_t^h = C_t^{h,j} = Y_t^j = C_t = Y_t = A_t L_t$ and $B_t^h = 0$. Productivity follows (2.4), the nominal rate follows (2.11) and symmetric prices, wages and transfers are

$$\begin{aligned} p_t^j &= p_t = p_0 \exp\left(\int_0^t \pi(q_s)ds\right), \\ w_t &= p_t C_t L_t^{m-1}, \quad D_t = p_t C_t (1 - L_t^m). \end{aligned} \quad (3.7)$$

Individual inflation is $\pi_t^j = \pi(q_t)$, and price-adjustment costs and profits are given by (2.6). Initial productivity and prices are positive constants.

Proposition 2 (Equilibrium dynamics). *The constructed equilibrium family is characterized by equations (3.3)—(3.6).*

- It satisfies the exact IS equation (2.16), the exact New Keynesian Phillips curve (2.15), and the policy rule (2.11).
- The output gap $\tilde{Y} = q - 1$ and inflation $\pi(q)$ are negative. Excess volatility σ_η^s is positive at every interior state $q \in (0, u_\eta)$.

¹⁰Therefore, aggregate volatility is countercyclical as in Lee and Dordal i Carreras (2026).

Proof. Equation (3.6) is already the IS equation (2.16) once $r_t - r^n = H(q_t)$, which itself is the Taylor rule (2.11). It remains only to verify the New Keynesian Phillips curve. Appendix A.3 evaluates the inflation generator after Lemma 1 establishes the state interval $q \in (0, u_\eta)$. $\square$

The way we construct this equilibrium family is simple: the volatility function $\sigma_\eta^s(q)$ is chosen so that the drift of the power inflation function $\pi(q)$ satisfies the non-linear Phillips curve (2.15). In the IS equation (2.16), precautionary saving offsets the consumption-growth effect of the negative real-rate deviation $H(q)$.

Excess volatility $\sigma_\eta^s(q)$ is the diffusion loading of the log output ratio. The loadings of the ratio itself and of inflation are $q\sigma_\eta^s(q)$ and $\beta\nu q^\nu\sigma_\eta^s(q)$. Both vanish as $q \downarrow 0$, while excess volatility tends to a positive finite limit.

3.1 The invariant distribution

For fixed $q_\circ \in (0, u_\eta)$, we define the integrated drift-to-variance ratio $\mathcal{R}_\eta$ and normalization N_η by

$$\mathcal{R}_\eta(q) = \int_{q_\circ}^q \frac{2\mu_\eta(\tilde{q})}{\zeta_\eta(\tilde{q})^2} d\tilde{q}, \quad (3.8)$$

and

$$N_\eta = \int_0^{u_\eta} \frac{e^{\mathcal{R}_\eta(q)}}{\zeta_\eta(q)^2} dq. \quad (3.9)$$

Dividing by N_η removes the dependence of the density on the arbitrary reference point $q_\circ$.

Proposition 3 (Non-degenerate stationary law). *The process (3.6) has a unique non-explosive strong solution from every interior initial state. Neither of the endpoints, 0 and u_η , is reached from the interior. It has a unique invariant density*

$$\varphi_{\eta,\star}(q) = \frac{e^{\mathcal{R}_\eta(q)}}{N_\eta \zeta_\eta(q)^2}, \quad 0 < q < u_\eta. \quad (3.10)$$

This density has full interior support $(0, u_\eta)$. From any interior initial state, the state process (3.6) converges in total variation to the distribution (3.10). Invariant initialization makes the output gap, inflation and excess volatility non-degenerate and stationary.

Proof. Appendix A.4 proves that the process never exits and that the distribution converges, using a Lyapunov drift bound and minorization on compact sets (Hairer and Mattingly, 2011). The scale and speed identities, the Lipschitz extension of the coefficients, and the Girsanov

comparison used in that proof are separately provided in Supplementary Appendices [S.1](#), [S.2](#), and [S.3](#). $\square$

In Appendix [A.4](#), we show that near $q \simeq 0$, the drift of $\log q$ is positive.¹¹ Near $q \simeq u_\eta$, the diffusion loading $\zeta_\eta(q)$ vanishes linearly in the distance to u_η and the drift points inward. We use these endpoint rates to classify the boundaries, and to prove that the process never exits and that invariant mass is finite. Finally, we prove that the process converges in total variation to the invariant density [\(3.10\)](#).

For an integrable function ψ , write $\mathbb{E}_\star[\psi(q)] = \int_0^{u_\eta} \psi(q) \varphi_{\eta,\star}(q) dq$. Inside such an expectation, q , $\tilde{Y} = q - 1$, $\pi = \pi(q)$, and $\sigma^s = \sigma_\eta^s(q)$ are all stationary random variables.

Invariant initialization draws q_0 from $\varphi_{\eta,\star}$ independently of future Brownian increments. In that case, the Phillips curve [\(2.15\)](#), IS equation [\(2.16\)](#), and Taylor rule [\(2.11\)](#) hold at every date, markets clear, and $(\tilde{Y}_t, \pi_t, \sigma_t^s)$ is strictly stationary. For fixed (b, ν) , Proposition [3](#) gives uniqueness of the invariant law of the process [\(3.6\)](#).

3.2 Properties of the stationary family

The equilibrium functions already constructed in [\(3.3\)](#), [\(3.4\)](#), and [\(3.6\)](#) determine the range of each observable and whether it rises or falls with the gap. Define the nominal policy-rate function $i(q) = r^n + \phi_y(q - 1) + \phi_\pi \pi(q)$ and the real-rate function $r(q) = r^n + H(q)$. The supremum of excess volatility σ_η^s is given by [\(3.11\)](#).

$$\bar{\sigma}^s = \frac{\sqrt{\sigma^2 + 2(1 + \nu)(\omega/\nu + \Phi) - \sigma}}{1 + \nu}. \quad (3.11)$$

Note that $\bar{\sigma}^s = \lim_{q \rightarrow 0} \sigma_\eta^s(q) = \sigma_\eta^s(0)$. For bounded stationary observables ψ_1 and ψ_2 , write $\text{Cov}_\star(\psi_1, \psi_2) = \mathbb{E}_\star[\psi_1 \psi_2] - \mathbb{E}_\star[\psi_1] \mathbb{E}_\star[\psi_2]$.

Proposition 4 (Bounds and cyclical comovement). *At every interior state, the equilibrium satisfies*

$$-1 < \tilde{Y} < u_\eta - 1 < 0, \quad \underline{\pi} < \pi < \pi(u_\eta) < 0, \quad (3.12)$$

$$0 < \sigma^s < \bar{\sigma}^s, \quad -\Phi < r - r^n < H(u_\eta) < 0. \quad (3.13)$$

Labor, marginal cost and transfers relative to nominal consumption are bounded, with

$$0 < L < L^n u_\eta, \quad 0 < \text{mc} < \alpha u_\eta^m, \quad 1 - \alpha u_\eta^m < \frac{D}{pC} < 1. \quad (3.14)$$

¹¹Itô's formula applied to [\(3.6\)](#) gives this drift as $\ell_\eta(q) = H(q) + \sigma \sigma_\eta^s(q) + \frac{1}{2}(\sigma_\eta^s(q))^2$.

Inflation and both interest rates strictly increase with the output gap, while excess volatility strictly decreases. Under the invariant law,

$$\begin{aligned} \text{Cov}_\star(\tilde{Y}, \pi) &> 0, \quad \text{Cov}_\star(\tilde{Y}, \sigma^s) < 0, \\ \text{Cov}_\star(\pi, \sigma^s) &< 0. \end{aligned} \tag{3.15}$$

Proof. Appendix B proves the endpoint bounds and derivative signs. Strict covariance signs follow from those derivatives and the invariant distribution's full interior support as claimed by Proposition 3. $\square$

Any interior state of every constructed equilibrium features underproduction, deflation, and a real rate below its natural counterpart, i.e., $H(q) = i(q) - \pi(q) - r^n < 0$. Inflation is procyclical in q and excess volatility is countercyclical. Both comovements compare states under the invariant distribution at a given date.

Frisch elasticity of labor supply The curvature of labor disutility (Frisch elasticity of labor supply) η moves the state interval $(0, u_\eta)$ without moving the inflation function itself in this class of stationary equilibria.

Corollary 1 (Labor elasticity at a common state). *Holding parameters (b, ν) fixed, a lower Frisch elasticity η raises the upper ratio endpoint u_η and raises excess volatility σ_η^s at every state $q \in (0, u_\eta)$. It leaves the inflation function $\pi(q)$ at a common state unchanged.*

Proof. If $\eta_1 < \eta_2$ ($m_1 > m_2$), $q^{m_1-\nu} < q^{m_2-\nu}$ for $0 < q < 1$. (3.4) gives $F_{\eta_1}(q) > F_{\eta_2}(q)$. Both $F_{\eta_1}(q)$ and $F_{\eta_2}(q)$ are decreasing in q due to Lemma 1, implying $u_{\eta_1} > u_{\eta_2}$, and (3.5) proves the proposition. Equation (3.3) has no dependence on η . $\square$

Thus, among this family of stationary equilibria, the Frisch elasticity η shifts the invariant distribution and its domain,¹² without moving the inflation function itself.

3.2.1 Expected gap and inflation

For any non-negative real exponent θ , define the invariant moment from (3.10):

$$M_{\eta, \theta} \equiv \mathbb{E}_\star[q^\theta] = \frac{\int_0^{u_\eta} q^\theta e^{\mathcal{R}_\eta(q)} \zeta_\eta(q)^{-2} dq}{\int_0^{u_\eta} e^{\mathcal{R}_\eta(q)} \zeta_\eta(q)^{-2} dq}. \tag{3.16}$$

Proposition 3 guarantees that $M_{\eta, \theta}$ is well-defined.

¹²It also changes the natural level of labor L^n . See Appendix A.2.

Proposition 5 (Stationary means and drift restrictions). *The stationary expected proportional gap and inflation are*

$$\mathbb{E}_\star[\tilde{Y}] = M_{\eta,1} - 1, \quad \mathbb{E}_\star[\pi] = \underline{\pi} + \beta M_{\eta,\nu}. \quad (3.17)$$

They are strictly negative and strictly between the corresponding endpoint bounds in (3.12). They satisfy the following exact restrictions:

$$\phi_y \mathbb{E}_\star[\tilde{Y}] + (\phi_\pi - 1) \mathbb{E}_\star[\pi] + \sigma \mathbb{E}_\star[\sigma^s] + \frac{1}{2} \mathbb{E}_\star[(\sigma^s)^2] = 0, \quad (3.18)$$

$$\rho \mathbb{E}_\star[\pi] - \kappa(M_{\eta,m} - 1) - \frac{\varepsilon - 1}{2} \mathbb{E}_\star[\pi^2] = 0. \quad (3.19)$$

In particular,

$$\mathbb{E}_\star[\tilde{Y}] = - \frac{(\phi_\pi - 1) \mathbb{E}_\star[\pi] + \sigma \mathbb{E}_\star[\sigma^s] + \frac{1}{2} \mathbb{E}_\star[(\sigma^s)^2]}{\phi_y}, \quad (3.20)$$

$$\mathbb{E}_\star[\pi] = \frac{\kappa}{\rho} (M_{\eta,m} - 1) + \frac{\varepsilon - 1}{2\rho} \mathbb{E}_\star[\pi^2], \quad (3.21)$$

$$\mathbb{E}_\star[r^n - r] = \sigma \mathbb{E}_\star[\sigma^s] + \frac{1}{2} \mathbb{E}_\star[(\sigma^s)^2] > 0. \quad (3.22)$$

Non-degeneracy of the invariant density (3.10) also proves:

$$\mathbb{E}_\star[\pi] < \underline{\pi} + \beta(1 + \mathbb{E}_\star[\tilde{Y}])^\nu. \quad (3.23)$$

Proof. Appendix A.4 proves moment integrability. The moment formulas follow by integrating (3.3), and strict concavity of q^ν gives (3.23). $\square$

3.2.2 Long-run distributions and sample paths

Proposition 6 (Convergence without stabilization). *From an arbitrary initial point $q_0 \in (0, u_\eta)$, expectations of bounded variables converge to their invariant expectations. In particular,*

$$\lim_{t \rightarrow \infty} \mathbb{E}_q[\tilde{Y}_t] = M_{\eta,1} - 1, \quad \lim_{t \rightarrow \infty} \mathbb{E}_q[\pi_t] = \underline{\pi} + \beta M_{\eta,\nu}. \quad (3.24)$$

For each bounded measurable state function ψ , its time average satisfies

$$\frac{1}{T} \int_0^T \psi(q_t) dt \longrightarrow \mathbb{E}_\star[\psi(q)] \quad \text{almost surely.} \quad (3.25)$$

Nevertheless,

$$\liminf_{t \rightarrow \infty} q_t = 0, \quad \limsup_{t \rightarrow \infty} q_t = u_\eta \quad \text{almost surely.} \quad (3.26)$$

Consequently, neither the output gap, inflation nor excess volatility has a pathwise limit.

Proof. Total-variation convergence in Proposition 3 implies convergence of bounded expectations. The ergodic theorem proved in Appendix A.4 gives (3.25). Non-exit properties then prove (3.26). $\square$

3.3 An occasionally binding effective lower bound

The stationary equilibrium family of Section 3 survives a lower bound on the nominal interest rate i_t . Let $i(q)$ denote the Taylor rule in (2.11), and i^{elb} be a constant effective lower bound.¹³ The effective policy rate would be given by

$$i_t = \max\{i^{\text{elb}}, i(q_t)\}. \quad (3.27)$$

Along the inflation function, $i'(q) = \phi_y + \phi_\pi \pi'(q) > 0$. Hence, any floor satisfying $i(0+) < i^{\text{elb}} < i(u_\eta)$ binds below a unique threshold $q_{\text{elb}} \in (0, u_\eta)$ and is slack above it.

The floor introduces a kink in the real-rate deviation $r_t - r^n$. Define

$$H_B(q) = i^{\text{elb}} - r^n - \pi(q), \quad (3.28)$$

$$H_{\text{elb}}(q) = \max\{H(q), H_B(q)\}. \quad (3.29)$$

with $H(q)$ defined in (3.3). The two functions $H(q)$ and $H_B(q)$ coincide at q_{elb} . Replacing H by H_{elb} in the formula for $F_\eta(q)$ in (3.4) gives

$$F_{\eta, \text{elb}}(q) = \frac{\omega}{\nu} - \frac{\kappa}{\beta\nu} q^{1+1/\eta-\nu} - \frac{(\varepsilon-1)\beta}{2\nu} q^\nu - H_{\text{elb}}(q). \quad (3.30)$$

Equivalently, $F_{\eta, \text{elb}} = \min\{F_\eta, F_B\}$, where F_B is obtained by replacing H with H_B .

We impose the following additional restrictions:

$$0 < \nu < \frac{\varepsilon-1}{2}, \quad i(0+) < i^{\text{elb}} < i(u_\eta). \quad (3.31)$$

¹³The zero lower bound is one such floor when $i(0+) < 0 < i(u_\eta)$.

Proposition 7 (Stationarity equilibrium under a effective lower bound). *Under the restrictions in (3.31), $F_{\eta,\text{elb}}$ is continuous, strictly decreasing, and positive on $(0, u_\eta)$ with $F_{\eta,\text{elb}}(u_\eta) = 0$.*

- *The constructed equilibrium family of Section 3 is still an equilibrium under the effective lower bound i^{elb} . In (3.3) and (3.4), H and F_η are replaced by H_{elb} and $F_{\eta,\text{elb}}$.*
- *Excess volatility $\sigma_{\eta,\text{elb}}^s$ follows (3.5), with $F_\eta(q)$ replaced by $F_{\eta,\text{elb}}(q)$. It is strictly positive on $(0, u_\eta)$. The boundary classification of Proposition 3 continues to hold, so both regimes have positive stationary probability.*

Proposition 7 follows by checking monotonicity on the two pieces. On (q_{elb}, u_η) , $F_{\eta,\text{elb}} = F_\eta$, which is strictly decreasing by Lemma 1. On the binding interval $(0, q_{\text{elb}})$,

$$F'_{\eta,\text{elb}}(q) = F'_B(q) = -\frac{\kappa(m-\nu)}{\beta\nu} q^{m-\nu-1} + \beta \left(\nu - \frac{\varepsilon-1}{2} \right) q^{\nu-1}. \quad (3.32)$$

Both terms of (3.32) are negative under (3.31). The two functions F_B and F_η coincide at the threshold q_{elb} . Appendix A.5 provides the details. The boundary classification is unchanged because the lower bound becomes slack near u_η , while near zero it raises the real-rate deviation $r_t - r^n$ relative to the unconstrained Taylor prescription, strengthening the drift of $\log q$ away from the origin.

Since firm optimality conditions contain no nominal interest rate, Proposition 2 continues to establish the non-linear New Keynesian Phillips curve for the equilibrium family constructed from $F_{\eta,\text{elb}}$. The IS equation holds with $r_t - r^n = H_{\text{elb}}(q_t)$. In sum, the effective lower bound changes the policy-rate schedule and introduces a kink, but it leaves the upper endpoint and the continuum of stationary equilibria indexed by ν intact.

In a rigid-price economy, our earlier work (Dordal i Carreras and Lee, 2026) also finds alternative zero-lower-bound equilibria driven by self-fulfilling volatility. If stabilization at a future date is certain, current excess volatility disappears and a unique equilibrium is selected. In the stationary economy constructed here, every future date repeats the same policy: the nominal rate follows the Taylor rule or binds at the effective lower bound, according to the current state. Under that recurrence, the stationary equilibria persist, and excess volatility remains strictly positive.

4 Conclusion

Our earlier paper (Lee and Dordal i Carreras, 2026) shows that a conventional Taylor rule need not uniquely determine the equilibrium of a New Keynesian model, even when it delivers

local determinacy. This paper asks whether the multiplicity is an artifact of the log-linearized Phillips curve or a more fundamental feature of the model. We show that it survives the exact non-linear pricing equation. Holding preferences, the demand block, and the policy rule fixed, we construct a continuum of stationary equilibria in which the economy persistently operates below potential, inflation remains negative, and consumption risk generates excess volatility. The multiplicity is therefore not a consequence of linearizing firms' pricing decisions: it persists when the supply side is treated fully non-linearly as well.

The continuum reflects more than differences in steady-state outcomes. Each equilibrium supports its own long-run distribution of economic activity and inflation, indexed by the shape of the inflation function. Although the economy continues to fluctuate, its distribution converges to an invariant density, and time averages converge almost surely. Individual paths, however, do not settle down: they approach both boundaries of the state space arbitrarily closely, infinitely often. The same multiplicity persists when monetary policy is constrained by an occasionally binding effective lower bound, including a zero lower bound under the condition specified above.

These results establish a continuum of stationary equilibria, not a complete characterization of all bounded equilibria. They also leave open whether the selection mechanisms developed in our earlier paper—either a behavioral friction or fiscal policy in a non-Ricardian environment—can select a unique equilibrium under the exact non-linear Phillips curve. Answering this question would help determine whether the equilibrium-selection results obtained under log-linearization extend to the fully non-linear economy.

A Equilibrium dynamics and stationarity

In this section, we prove the propositions in the main text. Appendix A.1 proves Proposition 1, deriving both the exact IS equation (2.16) and New Keynesian Phillips curve (2.15). Appendix A.2 characterizes the benchmark flexible price economy. Appendix A.3 proves Proposition 2. Appendix A.4 proves Proposition 3.

Appendix A.4 relies on several mathematical properties concerning the general scale and speed identities, the Lipschitz extension, and the Girsanov comparison. These are derived self-containedly in Supplementary Appendices S.1, S.2, and S.3, respectively

A.1 Proof of Proposition 1

Household optimization and IS equation. We derive the optimality conditions for an individual household before imposing symmetry based on variational arguments. A labor change δL_t^h , offset by $\delta C_t^h = (w_t/p_t)\delta L_t^h$ with unchanged wealth, gives the utility increment

$$\frac{\delta C_t^h}{C_t^h} - (L_t^h)^{m-1} \delta L_t^h = \left[\frac{w_t}{p_t C_t^h} - (L_t^h)^{m-1} \right] \delta L_t^h = 0.$$

proving the following intratemporal optimal condition:¹⁴

$$\frac{w_t}{p_t C_t^h} = (L_t^h)^{1/\eta}. \quad (\text{A.1})$$

For the intertemporal condition, we introduce δB_t^h for $t \in [0, T]$ be an adapted, bounded, absolutely continuous variation in nominal wealth, zero at the endpoints of its time support, so $\delta B_0^h = \delta B_T^h = 0$. Hold labor fixed, and set $\delta C_t^h = (i_t \delta B_t^h - \partial_t \delta B_t^h)/p_t$, so the household budget constraint (2.2) is not affected. integration by parts gives the utility variation

$$\mathbb{E} \int_0^T \xi_t^{N,h} (i_t \delta B_t^h - \partial_t \delta B_t^h) dt = \mathbb{E} \int_0^T \delta B_t^h (d\xi_t^{N,h} + i_t \xi_t^{N,h} dt) = 0,$$

proving

$$\mathbb{E}_t \left[\frac{d\xi_t^{N,h}}{\xi_t^{N,h}} \right] = -i_t dt. \quad (\text{A.2})$$

Equations (A.1) and (A.2) prove (2.13).

¹⁴Recall that $m = 1 + 1/\eta$.

If $dC_t^h/C_t^h = \mu_t^{C,h}dt + \sigma_t^{C,h}dZ_t$, Itô's formula gives

$$\frac{d\xi_t^{N,h}}{\xi_t^{N,h}} = [-\rho - \pi_t - \mu_t^{C,h} + (\sigma_t^{C,h})^2]dt - \sigma_t^{C,h}dZ_t.$$

Equating its drift to $-i_t$ gives (2.14).¹⁵

The IS equation instead follows from the quotient $q_t = C_t/Y_t^n$, with consumption volatility $\sigma_t^C = \sigma + \sigma_t^s$. The quadratic terms in its Itô rule are $\sigma^2 dt$ from natural-output variance and $\sigma\sigma_t^C dt$ from covariance, with the latter subtracted. Using the consumption Euler equation and $r^n = \rho + g - \sigma^2$ gives

$$\begin{aligned} \frac{dq_t}{q_t} &= \frac{dC_t}{C_t} - \frac{dY_t^n}{Y_t^n} + \left(\frac{dY_t^n}{Y_t^n} \right)^2 - \frac{dC_t}{C_t} \frac{dY_t^n}{Y_t^n} \\ &= [r_t - \rho + (\sigma_t^C)^2 - g + \sigma^2 - \sigma\sigma_t^C] dt + (\sigma_t^C - \sigma)dZ_t \\ &= [r_t - r^n + \sigma\sigma_t^s + (\sigma_t^s)^2] dt + \sigma_t^s dZ_t. \end{aligned} \tag{A.3}$$

Since $dq_t = d\tilde{Y}_t$, this is equation (2.16). □

Firm optimization and the exact New Keynesian Phillips curve. From (2.6), firm j 's real profit is given by

$$\frac{\Psi_s^j}{p_s} = Y_s \left[\left(\frac{p_s^j}{p_s} \right)^{1-\varepsilon} \left(1 - \frac{\tau(\pi_s^j)^2}{2} \right) - \text{mc}_s \left(\frac{p_s^j}{p_s} \right)^{-\varepsilon} \right].$$

This is the real-profit flow inside the firm's objective (2.7). The household definition $\xi_t = p_t \xi_t^N$ gives $\xi_t = e^{-\rho t}/C_t$. In the symmetric equilibrium, market clearing gives $Y_t = C_t$, so

$$\frac{\xi_s}{\xi_t} \frac{\Psi_s^j}{p_s} = e^{-\rho(s-t)} C_t \cdot \frac{\Psi_s^j}{p_s Y_s}.$$

Let $z_t^j = \log(p_t^j/p_t)$ denote firm j 's log relative price. Then

$$\frac{\Psi_s^j}{p_s Y_s} = f_s^z(z_s^j, \pi_s^j).$$

where

$$f_t^z(z, \varpi) = e^{-(\varepsilon-1)z} \left(1 - \frac{\tau\varpi^2}{2} \right) - \text{mc}_t e^{-\varepsilon z}. \tag{A.4}$$

The time subscript t on f_t^z records its dependence on aggregate marginal cost. The relative

¹⁵Note that this has been similarly proved in Lee and Dordal i Carreras (2026).

price satisfies $dz_t^j = (\pi_t^j - \pi_t)dt$.

For firm j , the control is π_t^j and the state is z_t^j . At generic state z , own inflation ϖ and costate λ_z , the normalized current-value Hamiltonian is $\mathcal{K}_t(z, \varpi, \lambda_z) = f_t^z(z, \varpi) + \lambda_z(\varpi - \pi_t)$. The first-order conditions before imposing firm symmetry give

$$\lambda_{z,t}^j = \tau e^{-(\varepsilon-1)z_t^j} \pi_t^j, \quad (\text{A.5})$$

and

$$d\lambda_{z,t}^j = \left[\rho \lambda_{z,t}^j - \partial_z f_t^z(z_t^j, \pi_t^j) \right] dt + dM_t^j. \quad (\text{A.6})$$

Here $\lambda_{z,t}^j$ is the relative-price costate and M_t^j is its local martingale component. Localize on finite horizons where the relative price stays in a compact subset of $(0, \infty)$ and the stochastic integrals below are true martingales.

From (A.6), we obtain

$$\partial_z f_t^z(z, \varpi) = -(\varepsilon - 1)e^{-(\varepsilon-1)z} + \varepsilon mc_t e^{-\varepsilon z} + \frac{(\varepsilon - 1)\tau}{2} e^{-(\varepsilon-1)z} \varpi^2. \quad (\text{A.7})$$

From (A.5), we have

$$d\lambda_{z,t}^j = \tau e^{-(\varepsilon-1)z_t^j} \left[d\pi_t^j - (\varepsilon - 1)(\pi_t^j - \pi_t) \pi_t^j dt \right].$$

Substituting this and (A.7) into (A.6) gives

$$d\pi_t^j = \left[\rho \pi_t^j + \frac{\varepsilon - 1}{\tau} - \frac{\varepsilon mc_t}{\tau} e^{-z_t^j} + \frac{\varepsilon - 1}{2} (\pi_t^j)^2 - (\varepsilon - 1) \pi_t \pi_t^j \right] dt + \frac{e^{(\varepsilon-1)z_t^j}}{\tau} dM_t^j. \quad (\text{A.8})$$

Setting $z_t^j = 0$, $\pi_t^j = \pi_t$ and $mc_t = \alpha q_t^m$ yields equation (2.15). $\square$

A.2 The flexible price economy

To obtain the natural allocation, note that for household j and firm j , consumption $C_t^{h,j}$ is proportional to $p_t^\varepsilon (p_t^j)^{-\varepsilon}$.

In the flexible-price problem, generic relative price $x = p_t^j/p_t > 0$ gives real profit divided by aggregate expenditure equal to $x^{1-\varepsilon} - mc_t x^{-\varepsilon}$. Its derivative is $x^{-\varepsilon-1}[(1-\varepsilon)x + \varepsilon mc_t]$, positive below mc_t/α and negative above it. Hence the individual static optimum is $x = mc_t/\alpha$.

Firm symmetry ($p_t^j = p_t$) yields $mc^n = \alpha$. Separately, imposing household symmetry on its intratemporal condition (A.1) and using $C_t = A_t L_t$ gives $mc_t = L_t^m$. Therefore $L^n = \alpha^{1/m}$,

$Y_t^n = A_t L^n$ and (2.10) follows from (2.8). Finally, natural consumption C_t^n has growth drift g and loading σ ; its Euler equation yields $r^n = \rho + g - \sigma^2$.

A.3 The construction domain

The constructed equilibrium family in Section 3 is defined on $(0, u_\eta)$. The next lemma checks that assumptions in (3.2) produce exactly one u_η , with negative inflation and positive excess volatility on the interval $(0, u_\eta)$.

Lemma 1 (Construction domain). *Under the assumptions in (3.2), $F_\eta(q)$ is strictly decreasing on $(0, 1)$ and has a unique zero $u_\eta \in (0, 1)$. Both $\pi(q)$ and $H(q)$ functions¹⁶ are strictly negative on $(0, u_\eta)$, and excess volatility $\sigma_\eta^s(q)$ is strictly positive on $(0, u_\eta)$.*

Proof. Differentiating $F_\eta(q)$ in (3.4) gives

$$F'_\eta(q) = -\frac{\kappa(m-\nu)}{\beta\nu}q^{m-\nu-1} - \frac{(\varepsilon-1)\beta}{2}q^{\nu-1} - \phi_y - (\phi_\pi - 1)\beta\nu q^{\nu-1} < 0. \quad (\text{A.9})$$

The endpoint values are $F_\eta(0+) = \omega/\nu + \Phi > 0$ and

$$F_\eta(1) = \pi(1) \left[\frac{\rho - (\varepsilon-1)\pi(1)/2}{\beta\nu} - (\phi_\pi - 1) \right].$$

Indeed, $\pi(1) = (1-b)\underline{\pi} < 0$ and $\beta = -b\underline{\pi}$ give the exact factorization

$$-\frac{b\nu}{1-b}F_\eta(1) = \rho - \frac{\varepsilon-1}{2}(1-b)\underline{\pi} - \beta\nu(\phi_\pi - 1) > 0.$$

Therefore, the last inequality in (3.2) is exactly equivalent to $F_\eta(1) < 0$. The root u_η exists uniquely and interior volatility is positive. For $0 < q < 1$, $\pi(q) < \pi(1) < 0$. Since $\phi_y > 0$ and $\phi_\pi \geq 1$, the gap term in $H(q)$ is strictly negative and the inflation term is non-positive, so $H(q) < 0$. $\square$

Proposition 2 asserts that (3.3), (3.4), (3.5), and (3.6) satisfy the exact New Keynesian Phillips curve (2.15) and the IS equation (2.16) on $(0, u_\eta)$.

Proof of Proposition 2. Equation (3.6) is already the IS equation (2.16) once $r_t - r^n = H(q_t)$, which itself is the Taylor rule (2.11). It remains only to verify the New Keynesian Phillips curve. The excess volatility $\sigma_\eta^s(q)$ satisfies $\sigma\sigma_\eta^s(q) + (1+\nu)(\sigma_\eta^s(q))^2/2 = F_\eta(q)$. Differentiating

¹⁶Recall that $H(q) = r_t - r^n$.

the power inflation function twice and applying Itô's formula gives

$$\begin{aligned}
\mathcal{L}_\eta \pi(q) &= \beta \nu q^{\nu-1} \mu_\eta(q) + \frac{\beta \nu (\nu-1)}{2} q^{\nu-2} \zeta_\eta(q)^2 \\
&= \beta \nu q^\nu \left[H(q) + \sigma \sigma_\eta^s(q) + \frac{1+\nu}{2} (\sigma_\eta^s(q))^2 \right] \\
&= \omega \beta q^\nu - \kappa q^m - \frac{\varepsilon-1}{2} \beta^2 q^{2\nu} \\
&= \rho \pi(q) - \kappa (q^m - 1) - \frac{\varepsilon-1}{2} \pi(q)^2,
\end{aligned} \tag{A.10}$$

implying that the New Keynesian Phillips curve (2.15) is satisfied. $\square$

A.4 Proof of Proposition 3

Proposition 3 asserts three things about the diffusion process (3.6): a unique strong solution exists for all time and never reaches either endpoint; the process has the invariant density (3.10); and the distribution converges to that density from every interior start. The proof proceeds in nine steps.

The first four concern a neighborhood of $q = 0$. A path cannot reach the origin ($q \sim 0$) in finite time. Occupation near the origin is still integrable though, so the contribution of $\log q$ to consumption utility has a finite invariant mean. An interior reference point is reached in finite expected time from its left. A path that starts arbitrarily close to the origin does not reach that reference point in finite expected time, so the origin is not an entrance boundary. The next three steps ask the same questions at $q = u_\eta$. The upper endpoint is inaccessible from the interior, the invariant mass near it is finite though, and it is an entrance boundary: a path starting arbitrarily close to u_η reaches the interior in finite expected time.

Those seven steps evaluate two standard integrals of a one-dimensional diffusion. Supplementary Appendix S.1 derives them. Scale distance decides whether one point is hit before another. Speed mass decides how much time is spent near an endpoint, and hence whether that tail contributes a finite amount to masses. A nested integral of the two decides whether the process can enter from an endpoint it cannot hit.

The eighth step extends the same coefficients Lipschitz-continuously up to both endpoints, so the state equation has a strong solution up to the first exit and needs no modification near the origin. That extension is provided in Supplementary Appendix S.2. The ninth step produces the pull toward the interior and the overlap of transition probabilities that turn non-exit into convergence. The change of coordinate and the Girsanov weight used for the overlap are in Supplementary Appendix S.3; the overlap constant itself is assembled back in

this proof.

Proof of Proposition 3. The coefficients μ_η and ζ_η are smooth with positive diffusion in the interior, giving local strong existence and uniqueness up to exit.

Let $\ell_\eta(q) = \mu_\eta(q)/q - (\sigma_\eta^s(q))^2/2$ denote the drift of $\log q$. Itô's formula on $\log q$ and (3.6) yield

$$\ell_\eta(q) = H(q) + \sigma\sigma_\eta^s(q) + \frac{1}{2}(\sigma_\eta^s(q))^2. \quad (\text{A.11})$$

The lower limit $\ell_0 = \lim_{q \downarrow 0} \ell_\eta(q)$ is obtained by evaluating each term at $q = 0+$. Inflation tends to $\underline{\pi}$, so

$$H(0+) = \phi_y(0-1) + (\phi_\pi - 1)\underline{\pi} = -\Phi$$

by the definition of Φ in (3.1). In (3.4) the two power terms vanish as $q \downarrow 0$, because $\nu > 0$ and $1 + 1/\eta - \nu > 0$, and therefore $F_\eta(0+) = \omega/\nu + \Phi$. Substituting this into (3.5) produces $\bar{\sigma}^s$ in (3.11). Hence

$$\ell_0 = -\Phi + \sigma\bar{\sigma}^s + \frac{1}{2}(\bar{\sigma}^s)^2.$$

Rationalizing (3.11) gives $(1+\nu)\bar{\sigma}^s + \sigma = \sqrt{\sigma^2 + 2(1+\nu)(\omega/\nu + \Phi)}$. Squaring and cancelling σ^2 yield

$$\sigma\bar{\sigma}^s + \frac{1+\nu}{2}(\bar{\sigma}^s)^2 = \frac{\omega}{\nu} + \Phi.$$

Subtract $\Phi + (\nu/2)(\bar{\sigma}^s)^2$ from both sides:

$$\ell_0 = \frac{\omega}{\nu} - \frac{\nu}{2}(\bar{\sigma}^s)^2.$$

Solve the same identity for $(\bar{\sigma}^s)^2$ and substitute once more:

$$\ell_0 = \frac{\omega - \nu^2\Phi}{\nu(1+\nu)} + \frac{\nu\sigma\bar{\sigma}^s}{1+\nu}. \quad (\text{A.12})$$

The assumption $\nu^2\Phi < \omega$ and $\bar{\sigma}^s > 0$ in (3.2) make the ℓ_0 strictly positive. Collected,

$$\begin{aligned} \bar{\sigma}^s &= \frac{\sqrt{\sigma^2 + 2(1+\nu)(\omega/\nu + \Phi)} - \sigma}{1+\nu} > 0, \\ \ell_0 &= \frac{\omega - \nu^2\Phi}{\nu(1+\nu)} + \frac{\nu\sigma\bar{\sigma}^s}{1+\nu} > 0. \end{aligned} \quad (\text{A.13})$$

Roughly speaking, the positive limit ℓ_0 is what makes the $q \sim 0$ point repelling. To turn it into the four conclusions listed above, form the scale integrand of Supplementary Appendix S.1.

From (3.6), $\zeta_\eta(q) = q\sigma_\eta^s(q)$ and $\mu_\eta(q)/q = \ell_\eta(q) + (\sigma_\eta^s(q))^2/2$, so the scale integrand is

$$\frac{2\mu_\eta(q)}{\zeta_\eta(q)^2} = \frac{2}{q(\sigma_\eta^s(q))^2} \left[\ell_\eta(q) + \frac{1}{2}(\sigma_\eta^s(q))^2 \right].$$

Send $q \downarrow 0$. The bracket tends to $\ell_0 + (\bar{\sigma}^s)^2/2$ and the remaining coefficient tends to $2/(\bar{\sigma}^s)^2$. Therefore

$$\frac{2\mu_\eta(q)}{\zeta_\eta(q)^2} \longrightarrow \frac{2\ell_0 + (\bar{\sigma}^s)^2}{q(\bar{\sigma}^s)^2} = \frac{1 + \gamma_0}{q}, \quad \gamma_0 \equiv \frac{2\ell_0}{(\bar{\sigma}^s)^2} > 0. \quad (\text{A.14})$$

The exponent γ_0 is exactly the extra unit of $1/q$ coming from the positive $\log q$ drift limit. From (3.3) and (3.1),

$$H(q) + \Phi = \phi_y q + (\phi_\pi - 1)\beta q^\nu.$$

Since $0 < \nu < 1$, this is $O(q^\nu)$. The two power terms in (3.4) are of orders $q^{m-\nu}$ and q^ν , with $m - \nu = 1 + 1/\eta - \nu > 0$ because $\nu < 1$. Also $H(q) - H(0+) = H(q) + \Phi$. Therefore

$$F_\eta(q) - F_\eta(0+) = O(q^\vartheta), \quad \vartheta = \min(\nu, m - \nu) > 0.$$

Differentiating (3.5) with respect F_η , we obtain

$$\frac{d\sigma_\eta^s}{dF_\eta} = \frac{1}{\sigma + (1 + \nu)\sigma_\eta^s}.$$

On $[0, u_\eta]$, $F_\eta > 0$, hence $\sigma_\eta^s > 0$. With $\sigma > 0$, $\sigma + (1 + \nu)\sigma_\eta^s \geq \sigma$ at every q , including its limit $q \sim 0$. Thus, the derivative becomes bounded by $1/\sigma$, uniformly in q and in F_η . Using the mean value theorem, there exists an intermediate value $F_\star(q)$ between $F_\eta(q)$ and $F_\eta(0+)$ such that¹⁷

$$\sigma_\eta^s(q) - \bar{\sigma}^s = \frac{F_\eta(q) - F_\eta(0+)}{\sigma + (1 + \nu)\sigma_\eta^s|_{F_\star(q)}}, \quad |\sigma_\eta^s(q) - \bar{\sigma}^s| \leq \frac{1}{\sigma} |F_\eta(q) - F_\eta(0+)|.$$

The numerator is $O(q^\vartheta)$, so $\sigma_\eta^s(q) - \bar{\sigma}^s = O(q^\vartheta)$.

The log-ratio drift is $\ell_\eta = H + \sigma\sigma_\eta^s + (\sigma_\eta^s)^2/2$. Subtract the limit $\ell_0 = -\Phi + \sigma\bar{\sigma}^s + (\bar{\sigma}^s)^2/2$:

$$\ell_\eta(q) - \ell_0 = (H(q) + \Phi) + \sigma(\sigma_\eta^s(q) - \bar{\sigma}^s) + \frac{1}{2}(\sigma_\eta^s(q) - \bar{\sigma}^s)(\sigma_\eta^s(q) + \bar{\sigma}^s).$$

The first term is $O(q^\nu)$ and $\vartheta \leq \nu$, so it is $O(q^\vartheta)$. The remaining factors σ and $\sigma_\eta^s + \bar{\sigma}^s$ stay

¹⁷The intermediate point $F_\star(q)$ depends on q , and so does the denominator evaluated there.

bounded near $q = 0$. Hence $\ell_\eta(q) - \ell_0 = O(q^\vartheta)$.

Now we return to scale in Supplementary Appendix S.1. From $\mu_\eta/q = \ell_\eta + (\sigma_\eta^s)^2/2$ and $\zeta_\eta = q\sigma_\eta^s$,

$$\frac{2\mu_\eta}{\zeta_\eta^2} = \frac{2\ell_\eta + (\sigma_\eta^s)^2}{q(\sigma_\eta^s)^2}, \quad \frac{1 + \gamma_0}{q} = \frac{2\ell_0 + (\bar{\sigma}^s)^2}{q(\bar{\sigma}^s)^2}.$$

Subtract:

$$\frac{2\mu_\eta(q)}{\zeta_\eta(q)^2} - \frac{1 + \gamma_0}{q} = \frac{1}{q} \left(\frac{2\ell_\eta(q) + (\sigma_\eta^s(q))^2}{(\sigma_\eta^s(q))^2} - \frac{2\ell_0 + (\bar{\sigma}^s)^2}{(\bar{\sigma}^s)^2} \right).$$

The parenthesis is the increment of $G(x, y) = (2x + y^2)/y^2 = 2x/y^2 + 1$ between $(\ell_\eta(q), \sigma_\eta^s(q))$ and $(\ell_0, \bar{\sigma}^s)$. Rewriting gives

$$G(\ell_\eta, \sigma_\eta^s) - G(\ell_0, \bar{\sigma}^s) = \frac{2[(\ell_\eta - \ell_0)(\bar{\sigma}^s)^2 + \ell_0(\bar{\sigma}^s - \sigma_\eta^s)(\bar{\sigma}^s + \sigma_\eta^s)]}{(\sigma_\eta^s)^2(\bar{\sigma}^s)^2}.$$

The two differences in the numerator are $O(q^\vartheta)$. The factors ℓ_0 , $\bar{\sigma}^s$ and $\bar{\sigma}^s + \sigma_\eta^s$ stay bounded near $q = 0$. As $\sigma_\eta^s(q) \rightarrow \bar{\sigma}^s > 0$, $\sigma_\eta^s(q) \geq \bar{\sigma}^s/2$ for all sufficiently small q , so the denominator is bounded below by a positive constant that does not depend on q .¹⁸ Hence the parenthesis is $O(q^\vartheta)$. Multiplication by $1/q$ produces $O(q^{\vartheta-1})$.

The scale density from (S.1.3) is

$$\mathfrak{s}'_\eta(q) = \exp \left[- \int_{q_0}^q \frac{2\mu_\eta(\tilde{q})}{\zeta_\eta(\tilde{q})^2} d\tilde{q} \right],$$

with the lower limit equal to q_0 so that $\mathfrak{s}'_\eta(q_0) = 1$. The lower tail uses $0 < q < q_0$. Hence

$$\mathfrak{s}'_\eta(q) = \exp \left[\int_q^{q_0} \frac{2\mu_\eta(\tilde{q})}{\zeta_\eta(\tilde{q})^2} d\tilde{q} \right].$$

Split the integrand into the limit term around $q \sim 0$ and a remainder:

$$\mathfrak{e}_\eta(\tilde{q}) = \frac{2\mu_\eta(\tilde{q})}{\zeta_\eta(\tilde{q})^2} - \frac{1 + \gamma_0}{\tilde{q}}.$$

We know from the above order calculation that $\mathfrak{e}_\eta(\tilde{q}) = O(\tilde{q}^{\vartheta-1})$ with the same $\vartheta = \min(\nu, m - \nu) > 0$. Then $\int_0^{q_0} |\mathfrak{e}_\eta|$ converges, because $\vartheta - 1 > -1$. Write

$$\int_q^{q_0} \frac{2\mu_\eta}{\zeta_\eta^2} d\tilde{q} = \int_q^{q_0} \frac{1 + \gamma_0}{\tilde{q}} d\tilde{q} + \int_q^{q_0} \mathfrak{e}_\eta(\tilde{q}) d\tilde{q}.$$

¹⁸The q -dependence of that denominator therefore changes only the implicit multiplicative constant.

The first integral is elementary:

$$\int_q^{q_\circ} \frac{1 + \gamma_0}{\tilde{q}} d\tilde{q} = (1 + \gamma_0) \log \frac{q_\circ}{q}.$$

The second integral tends to the finite value $\int_0^{q_\circ} \mathfrak{e}_\eta$ as $q \downarrow 0$, because $\int_0^q \mathfrak{e}_\eta \rightarrow 0$. Therefore

$$\mathfrak{s}'_\eta(q) = \exp \left((1 + \gamma_0) \log \frac{q_\circ}{q} + \int_q^{q_\circ} \mathfrak{e}_\eta \right) = \left(\frac{q_\circ}{q} \right)^{1+\gamma_0} \exp \left(\int_q^{q_\circ} \mathfrak{e}_\eta(\tilde{q}) d\tilde{q} \right).$$

Rewrite the exponential as $\exp(\int_0^{q_\circ} \mathfrak{e}_\eta) \exp(-\int_0^q \mathfrak{e}_\eta)$. Notice that the first factor is a positive constant. The second tends to 1 as $q \downarrow 0$. Collecting the powers of q gives

$$\mathfrak{s}'_\eta(q) = C_s q^{-1-\gamma_0} \exp \left(- \int_0^q \mathfrak{e}_\eta(\tilde{q}) d\tilde{q} \right), \quad (\text{A.15})$$

$$C_s = q_\circ^{1+\gamma_0} \exp \left(\int_0^{q_\circ} \mathfrak{e}_\eta(\tilde{q}) d\tilde{q} \right) > 0.$$

The exponential term in (A.15) tends to 1, so $\mathfrak{s}'_\eta(q) \sim C_s q^{-1-\gamma_0}$ around $q \sim 0$.

Speed limits can be calculated using (S.1.4). From (3.6), $\zeta_\eta(q) = q\sigma_\eta^s(q)$, hence $\zeta_\eta(q)^2 = q^2(\sigma_\eta^s(q))^2$. Thus

$$\mathfrak{m}_\eta(q) = \frac{2}{\zeta_\eta(q)^2 \mathfrak{s}'_\eta(q)} = \frac{2}{q^2(\sigma_\eta^s(q))^2} C_s^{-1} q^{1+\gamma_0} \exp \left(\int_0^q \mathfrak{e}_\eta \right).$$

As $q \downarrow 0$, $(\sigma_\eta^s(q))^2 \rightarrow (\bar{\sigma}^s)^2$ and the exponential tends to 1, so

$$\mathfrak{m}_\eta(q) \sim C_m q^{\gamma_0-1}, \quad C_m = \frac{2}{(\bar{\sigma}^s)^2 C_s} > 0.$$

Collected, as $q \downarrow 0$,

$$\mathfrak{s}'_\eta(q) \sim C_s q^{-1-\gamma_0}, \quad \mathfrak{m}_\eta(q) \sim C_m q^{\gamma_0-1}. \quad (\text{A.16})$$

Inaccessibility of $q = 0$. The first conclusion at the origin is that an interior path cannot reach it. Supplementary Appendix S.1 shows that this holds when the scale distance from an arbitrary interior point to 0 is infinite. With the normalization $\mathfrak{s}_\eta(q_\circ) = 0$, that distance is the elementary integral

$$\mathfrak{s}_\eta(q_\circ) - \mathfrak{s}_\eta(q) = \int_q^{q_\circ} \mathfrak{s}'_\eta(\tilde{q}) d\tilde{q}.$$

The equivalent $\mathfrak{s}'_\eta(\tilde{q}) \sim C_{\mathfrak{s}} \tilde{q}^{-1-\gamma_0}$ and $\gamma_0 > 0$ give

$$\int_q^{q_0} \mathfrak{s}'_\eta(\tilde{q}) d\tilde{q} \sim C_{\mathfrak{s}} \int_q^{q_0} \tilde{q}^{-1-\gamma_0} d\tilde{q} = \frac{C_{\mathfrak{s}}}{\gamma_0} (q^{-\gamma_0} - q_0^{-\gamma_0}) \sim \frac{C_{\mathfrak{s}}}{\gamma_0} q^{-\gamma_0}. \quad (\text{A.17})$$

Sending $q \downarrow 0$ sends this remaining scale to $+\infty$. Equivalently $\int_0^{q_0} \mathfrak{s}'_\eta = \infty$ and $\mathfrak{s}_\eta(q) \rightarrow -\infty$. By the hitting formula (S.1.2), the probability of reaching a neighborhood of 0 before a fixed interior barrier vanishes. The left endpoint is therefore inaccessible from the interior.

Integrable consumption utility near $q = 0$. The path never reaches 0, but consumption utility must remain integrable around the origin. Note that period utility contains $\log C = \log A + \log L^n + \log q$, and $\log q \rightarrow -\infty$ as $q \downarrow 0$.

On $[u_\eta/2, u_\eta]$ the logarithm is bounded, and only the left tail can destroy the moment. Supplementary Appendix S.1 shows that invariant density is proportional to speed mass m_η ,¹⁹ so the required integral is $\int_0^{q_0} |\log q| \mathfrak{m}_\eta(q) dq$.

First, speed mass itself is $\int_0^{q_0} \mathfrak{m}_\eta(q) dq$. $\mathfrak{m}_\eta(q) \sim C_{\mathfrak{m}} q^{\gamma_0-1}$ and $\gamma_0 > 0$ make $\gamma_0 - 1 > -1$, so the integral converges. Finite left-end speed mass is the contribution of $(0, q_0)$ to $N_\eta < \infty$: after normalization, that tail is a finite piece of invariant probability.

Because the invariant density is a normalization of speed, $\int |\log q| \mathfrak{m}_\eta$ is finite if and only if $\mathbb{E}_\star[|\log q|] < \infty$. The power $\gamma_0 - 1$ that made speed mass $\int_0^{q_0} \mathfrak{m}_\eta(q) dq$ finite can be shown to absorb this logarithm. To do that, note first that $|\log q| = o(q^{-\varepsilon})$ as $q \downarrow 0$ for any $\varepsilon > 0$.²⁰

Restrict further to $\varepsilon \in (0, \gamma_0)$ so that $|\log q| \mathfrak{m}_\eta(q)$ remains integrable: $\mathfrak{m}_\eta(q) \sim C_{\mathfrak{m}} q^{\gamma_0-1}$ and

$$|\log q| \mathfrak{m}_\eta(q) \sim C_{\mathfrak{m}} (|\log q| q^\varepsilon) q^{\gamma_0-1-\varepsilon}.$$

The parenthesis tends to zero, so it is bounded near $q = 0$, and therefore

$$|\log q| \mathfrak{m}_\eta(q) = O(q^{\gamma_0-1-\varepsilon})$$

with $\gamma_0 - 1 - \varepsilon > -1$. Therefore $\int_0^{q_0} |\log q| \mathfrak{m}_\eta(q) dq < \infty$.

Finite-time arrival from the left. Finite speed mass implies that time spent near 0 is integrable. Scale has also already ruled out hitting 0 from q_0 . We now go the other way: the

¹⁹It is given that scale diverges both at $q = 0$ and $q = u_\eta$, the latter of which is to be checked later.

²⁰Fix $\varepsilon > 0$ and set $t = -\log q$. Then $q \downarrow 0$ is $t \rightarrow +\infty$, $q^{-\varepsilon} = e^{\varepsilon t}$, and

$$\frac{|\log q|}{q^{-\varepsilon}} = \frac{t}{e^{\varepsilon t}} \rightarrow 0.$$

Hence $|\log q| q^\varepsilon \rightarrow 0$, which is $|\log q| = o(q^{-\varepsilon})$.

expected time to hit q_o starting from $q \in (0, q_o)$ to see whether a path that starts to the left of an interior point reaches that point in finite time. The limit of that expectation as $q \downarrow 0$ is then discussed after, where it decides whether the origin is an entrance. The expectation itself is the unique solution of the generator equation in the scale-speed form of Supplementary Appendix S.1.

First, the q -process cannot be started at 0, because 0 is not in the state space $(0, u_\eta)$. Then we place an auxiliary left barrier a with $0 < a < q < q_o$. We then compute the expected time to leave the bounded interval (a, q_o) starting from the interior point q ,²¹ and only afterwards send $a \downarrow 0$. Exit from (a, q_o) means the first time the state equals a or equals q_o . Hitting a is a move left toward 0. Hitting q_o is a move right to the initial point q . If we write $\mathfrak{h}_a \wedge \mathfrak{h}_{q_o}$ for that first exit time, we can define²²

$$\mathfrak{w}_a(q) = \mathbb{E}_q[\mathfrak{h}_a \wedge \mathfrak{h}_{q_o}]. \quad (\text{A.18})$$

Note that $\mathfrak{h}_a(q)$ is a function of the starting point q .

If the path starts already at a barrier, the exit time is zero, so $\mathfrak{w}_a(a) = \mathfrak{w}_a(q_o) = 0$. Apply (S.1.1) to $\mathfrak{w}_a$ along the path and stop at $\mathfrak{h}_a \wedge \mathfrak{h}_{q_o}$. On the compact interval $[a, q_o]$, the diffusion coefficient is bounded, and the stochastic integral becomes a true martingale and has mean zero. After stopping, $\mathfrak{w}_a$ equals zero at the exit point. Taking $\mathbb{E}_q$ therefore gives

$$0 = \mathfrak{w}_a(q) + \mathbb{E}_q \left[\int_0^{\mathfrak{h}_a \wedge \mathfrak{h}_{q_o}} \mathcal{L}_\eta \mathfrak{w}_a(q_s) ds \right]. \quad (\text{A.19})$$

Equation (A.19) holds if $\mathcal{L}_\eta \mathfrak{w}_a = -1$ from the definition in (A.18). Combined with the two boundary values, the generator problem

$$\mathcal{L}_\eta \mathfrak{w}_a = -1 \quad \text{on } (a, q_o), \quad \mathfrak{w}_a(a) = \mathfrak{w}_a(q_o) = 0$$

has a unique C^2 solution, and that solution is the expected exit time $\mathfrak{w}_a(q)$.

Supplementary Appendix A.4 shows that the scale-speed form gives $\mathcal{L}_\eta \psi = \mathfrak{m}_\eta^{-1} \partial_q(\psi' / \mathfrak{s}'_\eta)$.²³ Substituting $\psi = \mathfrak{w}_a$ and $\mathcal{L}_\eta \mathfrak{w}_a = -1$ therefore gives $\mathfrak{m}_\eta^{-1} \partial_q(\mathfrak{w}'_a / \mathfrak{s}'_\eta) = -1$, or

$$\partial_q \left(\frac{\mathfrak{w}'_a(q)}{\mathfrak{s}'_\eta(q)} \right) = -\mathfrak{m}_\eta(q).$$

²¹It then moves continuously inside (a, q_o) .

²²The expectation operator $\mathfrak{E}_q$ means that it is conditioned on the starting point q .

²³See equation (S.1.5) in Supplementary Appendix S.1.

Integrating from a to an interior point $x \in [a, q_\circ]$, we obtain

$$\frac{\mathfrak{w}'_a(x)}{\mathfrak{s}'_\eta(x)} - \frac{\mathfrak{w}'_a(a)}{\mathfrak{s}'_\eta(a)} = - \int_a^x \mathfrak{m}_\eta(y) dy.$$

Write K_a for the unknown constant $\mathfrak{w}'_a(a)/\mathfrak{s}'_\eta(a)$. Then

$$\mathfrak{w}'_a(x) = \mathfrak{s}'_\eta(x) \left(K_a - \int_a^x \mathfrak{m}_\eta(y) dy \right).$$

A second integration from a to q , using $\mathfrak{w}_a(a) = 0$, produces

$$\mathfrak{w}_a(q) = K_a \left(\mathfrak{s}_\eta(q) - \mathfrak{s}_\eta(a) \right) - \int_a^q \mathfrak{s}'_\eta(x) \left(\int_a^x \mathfrak{m}_\eta(y) dy \right) dx. \quad (\text{A.20})$$

The remaining constant is fixed by $\mathfrak{w}_a(q_\circ) = 0$:

$$K_a = \frac{\int_a^{q_\circ} \mathfrak{s}'_\eta(x) \left(\int_a^x \mathfrak{m}_\eta(y) dy \right) dx}{\mathfrak{s}_\eta(q_\circ) - \mathfrak{s}_\eta(a)}.$$

Using the integration by parts, we can rewrite the double integral as

$$\int_a^{q_\circ} \mathfrak{s}'_\eta(x) \left(\int_a^x \mathfrak{m}_\eta(y) dy \right) dx = \int_a^{q_\circ} \left(\mathfrak{s}_\eta(q_\circ) - \mathfrak{s}_\eta(y) \right) \mathfrak{m}_\eta(y) dy.$$

The same identity on $[a, q]$ replaces the second term in (A.20) by $\int_a^q \left(\mathfrak{s}_\eta(q) - \mathfrak{s}_\eta(y) \right) \mathfrak{m}_\eta(y) dy$.

Therefore

$$\mathfrak{w}_a(q) = \frac{\mathfrak{s}_\eta(q) - \mathfrak{s}_\eta(a)}{\mathfrak{s}_\eta(q_\circ) - \mathfrak{s}_\eta(a)} \int_a^{q_\circ} \left(\mathfrak{s}_\eta(q_\circ) - \mathfrak{s}_\eta(y) \right) \mathfrak{m}_\eta(y) dy - \int_a^q \left(\mathfrak{s}_\eta(q) - \mathfrak{s}_\eta(y) \right) \mathfrak{m}_\eta(y) dy. \quad (\text{A.21})$$

The first integral still runs over the whole of $(a, q_\circ)$. Split it at the starting point $y = q$:

$$\int_a^{q_\circ} \left(\mathfrak{s}_\eta(q_\circ) - \mathfrak{s}_\eta(y) \right) \mathfrak{m}_\eta(y) dy = \int_a^q \left(\mathfrak{s}_\eta(q_\circ) - \mathfrak{s}_\eta(y) \right) \mathfrak{m}_\eta(y) dy + \int_q^{q_\circ} \left(\mathfrak{s}_\eta(q_\circ) - \mathfrak{s}_\eta(y) \right) \mathfrak{m}_\eta(y) dy.$$

Substitute this split into (A.21). If we write $P_a(q) = (\mathfrak{s}_\eta(q) - \mathfrak{s}_\eta(a))/(\mathfrak{s}_\eta(q_\circ) - \mathfrak{s}_\eta(a))$ for the y -independent prefactor. Then

$$\begin{aligned} \mathfrak{w}_a(q) &= P_a(q) \int_a^q \left(\mathfrak{s}_\eta(q_\circ) - \mathfrak{s}_\eta(y) \right) \mathfrak{m}_\eta(y) dy + P_a(q) \int_q^{q_\circ} \left(\mathfrak{s}_\eta(q_\circ) - \mathfrak{s}_\eta(y) \right) \mathfrak{m}_\eta(y) dy \\ &\quad - \int_a^q \left(\mathfrak{s}_\eta(q) - \mathfrak{s}_\eta(y) \right) \mathfrak{m}_\eta(y) dy, \end{aligned} \quad (\text{A.22})$$

where the subtracted integral has upper limit q , not $q_\circ$. Note in (A.22) that the only surviving

piece on (q, q_0) is the middle integral, whose integrand is $P_a(q)(\mathfrak{s}_\eta(q_0) - \mathfrak{s}_\eta(y))\mathfrak{m}_\eta(y)$.

On (a, q) the first and third integrals are both present. The first supplies

$$P_a(q)(\mathfrak{s}_\eta(q_0) - \mathfrak{s}_\eta(y))\mathfrak{m}_\eta(y),$$

and the third subtracts $(\mathfrak{s}_\eta(q) - \mathfrak{s}_\eta(y))\mathfrak{m}_\eta(y)$. The net coefficient of $\mathfrak{m}_\eta(y)$ is therefore

$$\begin{aligned} & \frac{(\mathfrak{s}_\eta(q) - \mathfrak{s}_\eta(a))(\mathfrak{s}_\eta(q_0) - \mathfrak{s}_\eta(y))}{\mathfrak{s}_\eta(q_0) - \mathfrak{s}_\eta(a)} - (\mathfrak{s}_\eta(q) - \mathfrak{s}_\eta(y)) \\ &= \frac{1}{\mathfrak{s}_\eta(q_0) - \mathfrak{s}_\eta(a)} \left[(\mathfrak{s}_\eta(q) - \mathfrak{s}_\eta(a))(\mathfrak{s}_\eta(q_0) - \mathfrak{s}_\eta(y)) - (\mathfrak{s}_\eta(q) - \mathfrak{s}_\eta(y))(\mathfrak{s}_\eta(q_0) - \mathfrak{s}_\eta(a)) \right] \\ &= \frac{(\mathfrak{s}_\eta(q_0) - \mathfrak{s}_\eta(q))(\mathfrak{s}_\eta(y) - \mathfrak{s}_\eta(a))}{\mathfrak{s}_\eta(q_0) - \mathfrak{s}_\eta(a)}. \end{aligned}$$

The factor $\mathfrak{s}_\eta(q_0) - \mathfrak{s}_\eta(q)$ does not depend on y , so it leaves the integral. The contribution of (a, q) is therefore

$$\frac{\mathfrak{s}_\eta(q_0) - \mathfrak{s}_\eta(q)}{\mathfrak{s}_\eta(q_0) - \mathfrak{s}_\eta(a)} \int_a^q (\mathfrak{s}_\eta(y) - \mathfrak{s}_\eta(a))\mathfrak{m}_\eta(y) dy.$$

Adding the (q, q_0) contribution already recorded gives

$$\mathfrak{w}_a(q) = \frac{\mathfrak{s}_\eta(q_0) - \mathfrak{s}_\eta(q)}{\mathfrak{s}_\eta(q_0) - \mathfrak{s}_\eta(a)} \int_a^q (\mathfrak{s}_\eta(y) - \mathfrak{s}_\eta(a))\mathfrak{m}_\eta(y) dy + \frac{\mathfrak{s}_\eta(q) - \mathfrak{s}_\eta(a)}{\mathfrak{s}_\eta(q_0) - \mathfrak{s}_\eta(a)} \int_q^{q_0} (\mathfrak{s}_\eta(q_0) - \mathfrak{s}_\eta(y))\mathfrak{m}_\eta(y) dy. \quad (\text{A.23})$$

The two prefactors are the hitting probabilities from (S.1.2): $(\mathfrak{s}_\eta(q_0) - \mathfrak{s}_\eta(q))/(\mathfrak{s}_\eta(q_0) - \mathfrak{s}_\eta(a))$ is $\mathbb{P}_q(\mathfrak{h}_a < \mathfrak{h}_{q_0})$ and $(\mathfrak{s}_\eta(q) - \mathfrak{s}_\eta(a))/(\mathfrak{s}_\eta(q_0) - \mathfrak{s}_\eta(a))$ is $\mathbb{P}_q(\mathfrak{h}_{q_0} < \mathfrak{h}_a)$. The first integral runs over states to the left of the start, the second runs over states to the right. Each integrand is remaining scale from that side's absorbing barrier, weighted by speed.

Now send the auxiliary left barrier a to 0, still starting from the same interior q . We already know $\mathfrak{s}_\eta(a) \rightarrow -\infty$, which sends the probability of hitting a before q_0 to zero, so $\mathfrak{h}_a \wedge \mathfrak{h}_{q_0}$ increases to $\mathfrak{h}_{q_0}$. In the second prefactor $P_a(q) = (\mathfrak{s}_\eta(q) - \mathfrak{s}_\eta(a))/(\mathfrak{s}_\eta(q_0) - \mathfrak{s}_\eta(a))$, since both the numerator and the denominator are dominated by $-\mathfrak{s}_\eta(a) \rightarrow +\infty$, $P_a(q) \rightarrow 1$. The second integral does not depend on a , so the second term of $\mathfrak{w}_a(q)$ tends to $\int_q^{q_0} (\mathfrak{s}_\eta(q_0) - \mathfrak{s}_\eta(y))\mathfrak{m}_\eta(y) dy$.

The first term in (A.23) contains $\mathfrak{s}_\eta(a)$, meaning that we have to be careful in taking the limit $a \downarrow 0$. Write $\mathfrak{M}(z, q) = \int_z^q \mathfrak{m}_\eta(y) dy$ for the inner speed mass and apply the integration by parts to the first integral:

$$\int_a^q (\mathfrak{s}_\eta(y) - \mathfrak{s}_\eta(a))\mathfrak{m}_\eta(y) dy = \int_a^q \mathfrak{s}'_\eta(z) \mathfrak{M}(z, q) dz.$$

The identity $\mathfrak{s}_\eta(q) - \mathfrak{s}_\eta(a) = \int_a^q \mathfrak{s}'_\eta(z) dz$ makes the ratio of these two integrals the $\mathfrak{s}'_\eta$ -weighted average of $\mathfrak{M}(\cdot, q)$ on (a, q) . The first term of (A.23) is therefore

$$\frac{\mathfrak{s}_\eta(q_\circ) - \mathfrak{s}_\eta(q)}{\mathfrak{s}_\eta(q_\circ) - \mathfrak{s}_\eta(a)} \int_a^q \mathfrak{s}'_\eta(z) \mathfrak{M}(z, q) dz = \left(\mathfrak{s}_\eta(q_\circ) - \mathfrak{s}_\eta(q) \right) \frac{\int_a^q \mathfrak{s}'_\eta(z) \mathfrak{M}(z, q) dz}{\int_a^q \mathfrak{s}'_\eta(z) dz} P_a(q).$$

Finite left-end speed mass above implies $\mathfrak{M}(z, q) \rightarrow \int_0^q \mathfrak{m}_\eta$ as $z \downarrow 0$, and $0 \leq \mathfrak{M}(z, q) \leq \int_0^q \mathfrak{m}_\eta$ for every $z \in (a, q)$. Because $\mathfrak{s}'_\eta$ is not integrable at 0,²⁴ the probability measure with density $\mathfrak{s}'_\eta(z) / \int_a^q \mathfrak{s}'_\eta$ on (a, q) puts vanishing mass on any interval $[\delta, q]$ as $a \downarrow 0$. The weighted average thus sees only values of $\mathfrak{M}(z, q)$ with z near a , hence tends to $\int_0^q \mathfrak{m}_\eta$. Together with $P_a(q) \rightarrow 1$, the first term tends to $(\mathfrak{s}_\eta(q_\circ) - \mathfrak{s}_\eta(q)) \int_0^q \mathfrak{m}_\eta$. After the limit, $\mathfrak{s}'_\eta$ is no longer present: it was the weighting density, not a surviving factor. The left-hand side $\mathfrak{w}_a(q) = \mathbb{E}_q[\mathfrak{h}_a \wedge \mathfrak{h}_{q_\circ}]$ increases to $\mathbb{E}_q[\mathfrak{h}_{q_\circ}]$ by monotone convergence. Collecting the two limits yields

$$\mathbb{E}_q[\mathfrak{h}_{q_\circ}] = \left(\mathfrak{s}_\eta(q_\circ) - \mathfrak{s}_\eta(q) \right) \int_0^q \mathfrak{m}_\eta(y) dy + \int_q^{q_\circ} \left(\mathfrak{s}_\eta(q_\circ) - \mathfrak{s}_\eta(y) \right) \mathfrak{m}_\eta(y) dy. \quad (\text{A.24})$$

Both pieces are finite for each fixed $q \in (0, q_\circ)$: left-end speed mass is finite, remaining scale from q to $q_\circ$ is finite, and the second integrand is continuous on the compact interval $[q, q_\circ]$. Thus $q_\circ$ is reached in finite expected time from every interior start to its left.

The origin ($q = 0$) is not an entrance boundary. The above proof is about the expected time to go from a fixed interior start q to the interior reference $q_\circ$. We now solve the opposite question: can a path that begins arbitrarily close to 0 reach $q_\circ$ in finite expected time? The process cannot be initialized at $q = 0$, so the question is the limit

$$\lim_{q \downarrow 0} \mathbb{E}_q[\mathfrak{h}_{q_\circ}]. \quad (\text{A.25})$$

If it is finite, $q = 0$ is an entrance boundary: the state can come from 0 into the interior and hit $q_\circ$ in finite average time. If the limit is infinite, it cannot enter from 0 in finite time.²⁵

We send $q \downarrow 0$ (A.24). The first part is the product

$$\left(\mathfrak{s}_\eta(q_\circ) - \mathfrak{s}_\eta(q) \right) \int_0^q \mathfrak{m}_\eta(y) dy.$$

The remaining scale term $\mathfrak{s}_\eta(q_\circ) - \mathfrak{s}_\eta(q)$ was already computed in (A.17): $\mathfrak{s}_\eta(q_\circ) - \mathfrak{s}_\eta(q) =$

²⁴In (A.16), $-1 - \gamma_0 < -1$.

²⁵Note that this is not the earlier unattainability test, which asked whether a path starting at $q_\circ$ can hit 0.

$\int_q^{q_\circ} \mathfrak{s}'_\eta \sim (C_s/\gamma_0)q^{-\gamma_0}$. Finite speed mass near 0 and $\mathfrak{m}_\eta(y) \sim C_\mathfrak{m}y^{\gamma_0-1}$ give

$$\int_0^q \mathfrak{m}_\eta(y) dy \sim C_\mathfrak{m} \int_0^q y^{\gamma_0-1} dy = \frac{C_\mathfrak{m}}{\gamma_0} q^{\gamma_0}.$$

The product of these equivalents is

$$\frac{C_s}{\gamma_0} q^{-\gamma_0} \cdot \frac{C_\mathfrak{m}}{\gamma_0} q^{\gamma_0} = \frac{C_s C_\mathfrak{m}}{\gamma_0^2} = \frac{2}{\gamma_0^2 (\bar{\sigma}^s)^2},$$

using $C_\mathfrak{m} = 2/((\bar{\sigma}^s)^2 C_s)$. Thus the first piece in (A.24) tends to a finite positive constant. It cannot decide whether the limit of $\mathbb{E}_q[\mathfrak{h}_{q_\circ}]$ is finite.

The second piece is $\int_q^{q_\circ} (\mathfrak{s}_\eta(q_\circ) - \mathfrak{s}_\eta(y)) \mathfrak{m}_\eta(y) dy$. The integrand is non-negative. Sending $q \downarrow 0$ thus enlarges the domain of integration monotonically to $(0, q_\circ)$. Monotone convergence yields

$$\lim_{q \downarrow 0} \int_q^{q_\circ} (\mathfrak{s}_\eta(q_\circ) - \mathfrak{s}_\eta(y)) \mathfrak{m}_\eta(y) dy = \int_0^{q_\circ} (\mathfrak{s}_\eta(q_\circ) - \mathfrak{s}_\eta(y)) \mathfrak{m}_\eta(y) dy.$$

Using $\mathfrak{s}_\eta(q_\circ) - \mathfrak{s}_\eta(y) = \int_y^{q_\circ} \mathfrak{s}'_\eta$, the limit can be written as

$$\int_0^{q_\circ} \mathfrak{m}_\eta(y) \left(\int_y^{q_\circ} \mathfrak{s}'_\eta(\tilde{q}) d\tilde{q} \right) dy. \quad (\text{A.26})$$

The limit of the expected hitting time (A.25) is finite if and only if the nested integral (A.26) is finite, because the first piece stays bounded. If the nested integral diverges, then $\mathbb{E}_q[\mathfrak{h}_{q_\circ}] \rightarrow \infty$ as $q \downarrow 0$. Starting closer to 0 makes the expected clock from the left endpoint to $q_\circ$ arbitrarily long.

The inner integral is the remaining scale already computed in (A.17), which is asymptotic to a multiple of $y^{-\gamma_0}$. Multiplication by $\mathfrak{m}_\eta(y) \sim C_\mathfrak{m}y^{\gamma_0-1}$ cancels γ_0 in the exponent:

$$\mathfrak{m}_\eta(y) \int_y^{q_\circ} \mathfrak{s}'_\eta(\tilde{q}) d\tilde{q} \sim C_\mathfrak{m} \frac{C_s}{\gamma_0} y^{\gamma_0-1} y^{-\gamma_0} = \frac{2}{\gamma_0 (\bar{\sigma}^s)^2} y^{-1},$$

where the last equality uses $C_\mathfrak{m} = 2/((\bar{\sigma}^s)^2 C_s)$. The integral $\int_0^{q_\circ} y^{-1} dy$ diverges logarithmically, so the nested integral is infinite and $\mathbb{E}_q[\mathfrak{h}_{q_\circ}] \rightarrow \infty$ as $q \downarrow 0$.

The origin is therefore inaccessible from the interior, and the expected time to enter from it is infinite. In the boundary classification of Helland (1996, theorem 8), that pair of conclusions is a *natural* boundary.²⁶ Finite left-end speed mass is compatible with it: occupation near 0 can be integrable while the entrance clock is not.

²⁶In Lee and Dordal i Carreras (2026), we classify the stabilized path as a natural boundary in the case of martingale equilibria.

Now we show that at the upper endpoint $q = u_\eta$, the same nested integral converges, and it becomes an entrance rather than a natural boundary.

Inaccessibility of $q = u_\eta$. The same three questions discussed above are now asked around the upper end $q = u_\eta$, which is the unique root of F_η in $(0, 1)$ from Lemma 1. Around $q = u_\eta$, we write in local coordinates, i.e., the distance $w = u_\eta - q > 0$. As $q \uparrow u_\eta$ we check, in order, that scale distance from an arbitrary interior point to u_η is infinite, that speed mass near u_η is finite, and that the nested entrance integral from u_η down to q_o converges.

We begin with the volatility root (3.5). At $q = u_\eta$ one has $F_\eta(u_\eta) = 0$, so $\sigma_\eta^s(u_\eta) = 0$. Rationalize the numerator, we obtain

$$\sigma_\eta^s(q) = \frac{2F_\eta(q)}{\sqrt{\sigma^2 + 2(1 + \nu)F_\eta(q) + \sigma}}. \quad (\text{A.27})$$

The mean value theorem supplies $q_\star \in (q, u_\eta)$ with $F_\eta(q) = F'_\eta(q_\star)(q - u_\eta) = -F'_\eta(q_\star)(u_\eta - q)$. Lemma 1 proves $F'_\eta < 0$, so $-F'_\eta(q_\star) > 0$. As $q \uparrow u_\eta$, the intermediate point $q_\star$ tends to u_η and the denominator tends to $\sigma + \sigma = 2\sigma$. Therefore

$$\sigma_\eta^s(q) \sim \frac{-F'_\eta(u_\eta)}{\sigma}(u_\eta - q) = \varsigma_u(u_\eta - q), \quad \varsigma_u = -\frac{F'_\eta(u_\eta)}{\sigma} > 0.$$

The diffusion loading in (3.6) is $\zeta_\eta(q) = q\sigma_\eta^s(q)$. Since $q \rightarrow u_\eta$,

$$\zeta_\eta(q) \sim u_\eta \varsigma_u(u_\eta - q). \quad (\text{A.28})$$

The drift is $\mu_\eta(q) = q[H(q) + \sigma\sigma_\eta^s(q) + (\sigma_\eta^s(q))^2]$. Excess volatility $\sigma_\eta^s(q)$ vanishes at the root, and $q \rightarrow u_\eta$, so

$$\mu_\eta(q) \longrightarrow \mu_u = u_\eta H(u_\eta). \quad (\text{A.29})$$

Lemma 1 proves $H < 0$ on $(0, 1)$ and $u_\eta \in (0, 1)$, hence $\mu_u < 0$: near u_η , the drift points back into the interval.

The scale integrand is $2\mu_\eta/\zeta_\eta^2$. Write $w = u_\eta - q$. The functions F_η , H and the excess volatility in (A.27) are C^2 near u_η inside $(0, 1)$. $F_\eta(u_\eta) = 0$ give $F_\eta(q) = F'_\eta(u_\eta)(q - u_\eta) + O(w^2)$. With $\varsigma_u = -F'_\eta(u_\eta)/\sigma$ and $q - u_\eta = -w$, we have $F_\eta(q) = \sigma\varsigma_u w + O(w^2)$, meaning $F_\eta(q) =$

$O(w)$. The denominator in (A.27) therefore starts at σ :

$$\begin{aligned}\sqrt{\sigma^2 + 2(1 + \nu)F_\eta(q)} &= \sigma \sqrt{1 + \frac{2(1 + \nu)F_\eta(q)}{\sigma^2}} \\ &= \sigma + \frac{(1 + \nu)F_\eta(q)}{\sigma} + O(w^2) \\ &= \sigma + O(w).\end{aligned}$$

Adding σ gives $\sqrt{\sigma^2 + 2(1 + \nu)F_\eta} + \sigma = 2\sigma + O(w)$. At $w = 0$ this is exactly 2σ , and the error is the order of w . Substitute into (A.27):

$$\sigma_\eta^s(q) = \frac{2F_\eta(q)}{\sqrt{\sigma^2 + 2(1 + \nu)F_\eta(q)} + \sigma} = \frac{2(\sigma\zeta_u w + O(w^2))}{2\sigma + O(w)} = \zeta_u w + O(w^2).$$

With this, $\zeta_\eta(q) = q\sigma_\eta^s(q)$ is given by

$$\zeta_\eta(q) = (u_\eta - w)(\zeta_u w + O(w^2)) = u_\eta \zeta_u w + O(w^2),$$

which is consistent with (A.28). Squaring gives

$$\zeta_\eta(q)^2 = u_\eta^2 \zeta_u^2 w^2 (1 + O(w)),$$

and $H(q) = H(u_\eta) + O(w)$, $\sigma\sigma_\eta^s = O(w)$, $(\sigma_\eta^s)^2 = O(w^2)$, so

$$\mu_\eta(q) = (u_\eta - w)[H(u_\eta) + O(w)] = \mu_u + O(w),$$

which is consistent with (A.29). Therefore

$$\frac{2\mu_\eta(q)}{\zeta_\eta(q)^2} = \frac{2(\mu_u + O(w))}{u_\eta^2 \zeta_u^2 w^2 (1 + O(w))} = -\frac{C_u}{w^2} + O(w^{-1}), \quad C_u = -\frac{2\mu_u}{u_\eta^2 \zeta_u^2} > 0.$$

Since $2\mu_\eta(q)/\zeta_\eta(q)^2 + C_u/w^2 = O(w^{-1})$, we must have λ_u satisfying

$$\lambda_u = \lim_{q \uparrow u_\eta} (u_\eta - q) \left(\frac{2\mu_\eta(q)}{\zeta_\eta(q)^2} + \frac{C_u}{(u_\eta - q)^2} \right) = \lim_{w \downarrow 0} \frac{2\mu_\eta(q) w^2 + C_u \zeta_\eta(q)^2}{\zeta_\eta(q)^2 w},$$

meaning that $2\mu_\eta(q)/\zeta_\eta(q)^2$ around $q \sim u_\eta$ can now be written as

$$\frac{2\mu_\eta(q)}{\zeta_\eta(q)^2} = -\frac{C_u}{(u_\eta - q)^2} + \frac{\lambda_u}{u_\eta - q} + \rho_u(q), \tag{A.30}$$

where $\rho_u(q)$ is small, bounded on a left neighborhood of u_η .

For $q > q_\circ$ the scale density is

$$\mathfrak{s}'_\eta(q) = \exp \left[- \int_{q_\circ}^q \frac{2\mu_\eta(\tilde{q})}{\zeta_\eta(\tilde{q})^2} d\tilde{q} \right].$$

Fix an interior point $q_1 \in (q_\circ, u_\eta)$. Because the integral from $q_\circ$ to q_1 is a constant, we consider the integral from q_1 to q . On (q_1, q) , substituting (A.30), we obtain

$$- \int_{q_1}^q \frac{2\mu_\eta}{\zeta_\eta^2} d\tilde{q} = C_u \int_{q_1}^q \frac{d\tilde{q}}{(u_\eta - \tilde{q})^2} - \lambda_u \int_{q_1}^q \frac{d\tilde{q}}{u_\eta - \tilde{q}} - \int_{q_1}^q \rho_u(\tilde{q}) d\tilde{q}.$$

The first integral is $1/(u_\eta - q) - 1/(u_\eta - q_1)$. The second becomes $-\log(u_\eta - q) + \log(u_\eta - q_1)$. The third converges to a finite limit as $q \uparrow u_\eta$ because ρ_u is bounded. Therefore

$$- \int_{q_1}^q \frac{2\mu_\eta}{\zeta_\eta^2} d\tilde{q} = \frac{C_u}{u_\eta - q} + \lambda_u \log(u_\eta - q) + K_u(q),$$

where $K_u(q)$ tends to a finite limit $K_u(u_\eta -)$. Exponentiating gives

$$\mathfrak{s}'_\eta(q) = C_{s,u}(q) (u_\eta - q)^{\lambda_u} \exp \left[\frac{C_u}{u_\eta - q} \right],$$

where

$$C_{s,u}(q) = \exp \left(- \int_{q_\circ}^{q_1} \frac{2\mu_\eta}{\zeta_\eta^2} d\tilde{q} + K_u(q) \right),$$

which converges to a finite limit $C_{s,u} > 0$ as $q \uparrow u_\eta$.

Observe that

$$\int^{u_\eta} \exp \left[\frac{C_u}{u_\eta - q} \right] dq = \int_0^\infty \exp \left[\frac{C_u}{w} \right] dw = \int_0^\infty e^{C_u t} t^{-2} dt = \infty,$$

and the exponential dominates the power $(u_\eta - q)^{\lambda_u}$. Therefore $\int_{q_\circ}^{u_\eta} \mathfrak{s}'_\eta = \infty$ and $\mathfrak{s}_\eta(q) \rightarrow +\infty$ as $q \uparrow u_\eta$. By (S.1.2), the probability of hitting a neighborhood of u_η before a fixed interior barrier vanishes. The upper endpoint is inaccessible from the interior.

Finite invariant mass near $q = u_\eta$. Infinite scale distance has just shown that u_η is not accessible from the interior. The contribution of a neighborhood of u_η to the normalizing constant N_η is the speed mass of that neighborhood, again by Supplementary Appendix S.1.

Here, we show that this tail is finite. Speed is $\mathbf{m}_\eta = 2/(\zeta_\eta^2 \mathbf{s}'_\eta)$. Insert $\zeta_\eta^2 \sim u_\eta^2 \zeta_u^2 (u_\eta - q)^2$:

$$\mathbf{m}_\eta(q) \sim C_{\mathbf{m},u} (u_\eta - q)^{-2-\lambda_u} \exp \left[-\frac{C_u}{u_\eta - q} \right], \quad C_{\mathbf{m},u} = \frac{2}{u_\eta^2 \zeta_u^2 C_{\mathbf{s},u}} > 0.$$

The exponential now decays as $q \uparrow u_\eta$. With $w = u_\eta - \tilde{q}$, $dw = -d\tilde{q}$:

$$\int_q^{u_\eta} \mathbf{m}_\eta(\tilde{q}) d\tilde{q} \sim C_{\mathbf{m},u} \int_0^{u_\eta - q} w^{-2-\lambda_u} \exp \left[-\frac{C_u}{w} \right] dw. \quad (\text{A.31})$$

Differentiate $\Phi(w) = w^{-\lambda_u} \exp(-C_u/w)$,

$$\Phi'(w) = (C_u - \lambda_u w) w^{-2-\lambda_u} e^{-C_u/w}.$$

For small w one has $C_u - \lambda_u w \neq 0$, so

$$w^{-2-\lambda_u} e^{-C_u/w} = \frac{\Phi'(w)}{C_u - \lambda_u w} = \frac{\Phi'(w)}{C_u} \cdot \frac{1}{1 - (\lambda_u/C_u)w} = \frac{\Phi'(w)}{C_u} (1 + O(w)).$$

The integral in (A.31) from 0 to $W = u_\eta - q$ thus becomes

$$\int_0^W w^{-2-\lambda_u} e^{-C_u/w} dw = \frac{1}{C_u} \int_0^W \Phi'(w) dw + \int_0^W \Phi'(w) O(w) dw.$$

The first integral is $\Phi(W) - \Phi(0+)$. As $w \downarrow 0$, $\exp(-C_u/w)$ dominates every power of w , so $\Phi(0+) = 0$. The second integral satisfies

$$\left| \int_0^W \Phi'(w) O(w) dw \right| \leq K \int_0^W w \Phi'(w) dw \leq KW \int_0^W \Phi'(w) dw = KW \Phi(W)$$

for some finite K , because $\Phi' > 0$ for all sufficiently small w . Therefore

$$\int_0^W w^{-2-\lambda_u} e^{-C_u/w} dw = \frac{\Phi(W)}{C_u} (1 + O(W)) = \frac{W^{-\lambda_u}}{C_u} \exp \left(-\frac{C_u}{W} \right) (1 + O(W)).$$

The upper limit $W = u_\eta - q$ therefore gives

$$\int_q^{u_\eta} \mathbf{m}_\eta(\tilde{q}) d\tilde{q} \sim C_{u,\star} (u_\eta - q)^{-\lambda_u} \exp \left[-\frac{C_u}{u_\eta - q} \right], \quad C_{u,\star} = \frac{C_{\mathbf{m},u}}{C_u} > 0. \quad (\text{A.32})$$

In particular the tail is finite for every $q < u_\eta$, so the upper-end contribution to N_η is finite.

The upper endpoint is an entrance boundary. Inaccessibility and finite speed mass still leave open whether a path can be thought of as entering from u_η . As we have shown for the origin ($q = 0$), the entrance clock is the expected time to reach the interior reference $q_\circ$, now starting from $q \in (q_\circ, u_\eta)$, and then the limit of that expectation as $q \uparrow u_\eta$. A finite limit means u_η is an entrance. Similar to (A.24) and based on the scale-speed representation in Supplementary Appendix S.1, we can show by sending the right barrier $b \uparrow u_\eta$ and passing to the limit that

$$\mathbb{E}_q[\mathfrak{h}_{q_\circ}] = (\mathfrak{s}_\eta(q) - \mathfrak{s}_\eta(q_\circ)) \int_q^{u_\eta} \mathfrak{m}_\eta(y) dy + \int_{q_\circ}^q (\mathfrak{s}_\eta(y) - \mathfrak{s}_\eta(q_\circ)) \mathfrak{m}_\eta(y) dy. \quad (\text{A.33})$$

The first piece is remaining scale from $q_\circ$ to q times the speed tail from q to u_η . With

$$\mathfrak{s}_\eta(q) - \mathfrak{s}_\eta(q_\circ) = \int_{q_\circ}^q \mathfrak{s}'_\eta(\tilde{q}) d\tilde{q},$$

we fix $q_1 \in (q_\circ, u_\eta)$ as above. The integral from $q_\circ$ to q_1 is a finite constant. On (q_1, q) we use $\mathfrak{s}'_\eta(\tilde{q}) \sim C_{\mathfrak{s},u}(u_\eta - \tilde{q})^{\lambda_u} \exp[C_u/(u_\eta - \tilde{q})]$ and $w = u_\eta - \tilde{q}$, to obtain

$$\int_{q_1}^q \mathfrak{s}'_\eta(\tilde{q}) d\tilde{q} \sim C_{\mathfrak{s},u} \int_{u_\eta-q}^{u_\eta-q_1} w^{\lambda_u} \exp\left[\frac{C_u}{w}\right] dw.$$

The integrand explodes as $w \downarrow 0$. Set $\Xi(w) = w^{\lambda_u+2} \exp(C_u/w)$. Then

$$\Xi'(w) = w^{\lambda_u} \exp\left[\frac{C_u}{w}\right] ((\lambda_u + 2)w - C_u),$$

hence $w^{\lambda_u} \exp(C_u/w) = \Xi'(w)/((\lambda_u + 2)w - C_u)$. For small w the denominator is $-C_u + O(w)$, and the definite integral from $W = u_\eta - q$ to the fixed $u_\eta - q_1$ is therefore

$$\int_W^{u_\eta-q_1} w^{\lambda_u} e^{C_u/w} dw = \frac{\Xi(W)}{C_u} (1 + O(W)) = \frac{W^{\lambda_u+2}}{C_u} \exp\left[\frac{C_u}{W}\right] (1 + O(W))$$

up to a W -independent constant from the upper limit. Remaining scale is therefore

$$\mathfrak{s}_\eta(q) - \mathfrak{s}_\eta(q_\circ) \sim \frac{C_{\mathfrak{s},u}}{C_u} (u_\eta - q)^{\lambda_u+2} \exp\left[\frac{C_u}{u_\eta - q}\right].$$

The speed tail already computed in (A.32) is

$$\int_q^{u_\eta} \mathfrak{m}_\eta(y) dy \sim C_{u,\star} (u_\eta - q)^{-\lambda_u} \exp\left[-\frac{C_u}{u_\eta - q}\right].$$

The product of these two equivalents is

$$\left(\mathfrak{s}_\eta(q) - \mathfrak{s}_\eta(q_\circ)\right) \int_q^{u_\eta} \mathfrak{m}_\eta \sim \frac{C_{\mathfrak{s},u} C_{u,\star}}{C_u} (u_\eta - q)^2 \longrightarrow 0.$$

The constant piece of remaining scale, times the same tail, is $O((u_\eta - q)^{-\lambda_u} \exp[-C_u/(u_\eta - q)])$ and also tends to 0. Thus the first piece of $\mathbb{E}_q[\mathfrak{h}_{q_\circ}]$ tends to 0.

The second piece is $\int_{q_\circ}^q (\mathfrak{s}_\eta(y) - \mathfrak{s}_\eta(q_\circ)) \mathfrak{m}_\eta(y) dy$. The integrand is non-negative. Sending $q \uparrow u_\eta$ enlarges the domain monotonically, so by monotone convergence,

$$\lim_{q \uparrow u_\eta} \int_{q_\circ}^q (\mathfrak{s}_\eta(y) - \mathfrak{s}_\eta(q_\circ)) \mathfrak{m}_\eta(y) dy = \int_{q_\circ}^{u_\eta} (\mathfrak{s}_\eta(y) - \mathfrak{s}_\eta(q_\circ)) \mathfrak{m}_\eta(y) dy. \quad (\text{A.34})$$

Write $\mathfrak{s}_\eta(y) - \mathfrak{s}_\eta(q_\circ) = \int_{q_\circ}^y \mathfrak{s}'_\eta(z) dz$ and apply the integration by parts,

$$\int_{q_\circ}^{u_\eta} \mathfrak{m}_\eta(y) \left(\int_{q_\circ}^y \mathfrak{s}'_\eta(z) dz \right) dy = \int_{q_\circ}^{u_\eta} \mathfrak{s}'_\eta(z) \left(\int_z^{u_\eta} \mathfrak{m}_\eta(y) dy \right) dz.$$

The inner tail $\int_z^{u_\eta} \mathfrak{m}_\eta(y) dy$ is asymptotic to $C_{u,\star} (u_\eta - z)^{-\lambda_u} \exp[-C_u/(u_\eta - z)]$. Multiplication by the scale equivalent $\mathfrak{s}'_\eta(z) \sim C_{\mathfrak{s},u} (u_\eta - z)^{\lambda_u} \exp[+C_u/(u_\eta - z)]$ cancels the two exponentials and the two powers λ_u :

$$\mathfrak{s}'_\eta(z) \int_z^{u_\eta} \mathfrak{m}_\eta \longrightarrow C_{\mathfrak{s},u} C_{u,\star},$$

which is bounded and integrable on a neighborhood of u_η . Hence the integral (A.34) converges. Combined with the vanishing first piece, $\lim_{q \uparrow u_\eta} \mathbb{E}_q[\mathfrak{h}_{q_\circ}]$ is finite: a path that starts arbitrarily close to u_η reaches $q_\circ$ in finite expected time. Together with inaccessibility from the interior and finite speed mass, u_η is an entrance boundary. Neither endpoint is accessible from the interior.

The same two scale integrals also fix the invariant density. Supplementary Appendix S.1 shows that every non-negative stationary density of this diffusion has the form $\varphi = \mathfrak{m}_\eta(C_{\text{stat}} - \mathfrak{j} \mathfrak{s}_\eta)$. The constant $\mathfrak{j}$ is a probability current: the rate at which probability flows past a point. With $s_\eta(0^+) = -\infty$ and $s_\eta(u_\eta^-) = \infty$, the only viable current $\mathfrak{j}$ is $\mathfrak{j} = 0$, implying $\varphi = C_{\text{stat}} \mathfrak{m}_\eta$ with $C_{\text{stat}} > 0$.

Lipschitz coefficients and a strong solution. The above integrals around endpoints (0 and u_η) show that neither boundary is hit from the interior. A strong solution on the whole state interval $(0, u_\eta)$ still requires the coefficients $\mu_\eta(q)$ and $\zeta_\eta(q)$ to remain Lipschitz as those boundaries are approached, so that the process does not have to be altered or truncated.

The only possible obstruction is $q \downarrow 0$. Supplementary Appendix S.2 shows that the extra factor of q in $\mu_\eta(q)$ and $\zeta_\eta(q)$ expressions (3.6) restores bounded derivatives at the origin, and

therefore that μ_η and ζ_η extend Lipschitz-continuously to $[0, u_\eta]$. Local strong existence and uniqueness therefore run up to exit, even if the exit time is infinite.

Recurrence, Lyapunov control and ergodicity. The above endpoint integrals classify the two boundaries and, find the invariant density.²⁷ In order to finish the proof of Proposition 3, we need to prove that the process produces a drift that pulls the q -state back from both tails, and the overlap of transition laws makes the distribution forget its starting point. Those two ingredients are constructed here. Together, they give a unique non-explosive strong solution for all time, convergence in total variation from every interior start. The overlap part relies on the fundamental results of Supplementary Appendix S.3.

A Lyapunov function $\mathcal{V}(q_t)$ is a scalar test function of the state q_t that becomes large at both endpoints and whose generator is negative when $\mathcal{V}(q_t)$ is large. If such a function exists, $\mathcal{V}(q_t)$ itself is likely to fall when the q -path is near 0 or u_η .²⁸ The process is effectively pulled back from the tails on average, which is an input for global non-explosion.

Think about this candidate Lyapunov function:

$$\mathcal{V}_{\delta,k}(q) = q^{-\delta} + (u_\eta - q)^{-k}. \quad (\text{A.35})$$

Both summands are continuous, and strictly positive on $(0, u_\eta)$. As $q \downarrow 0$, $q^{-\delta} \rightarrow +\infty$. As $q \uparrow u_\eta$, $(u_\eta - q)^{-k} \rightarrow +\infty$. A continuous function on an open interval that diverges at both endpoints attains a global minimum at an interior point. Pick any $q_\star \in (0, u_\eta)$ and choose $\varepsilon > 0$ small enough that $\mathcal{V}_{\delta,k} > \mathcal{V}_{\delta,k}(q_\star)$ on $(0, \varepsilon] \cup [u_\eta - \varepsilon, u_\eta)$. The restriction of $\mathcal{V}_{\delta,k}$ to the compact interval $[\varepsilon, u_\eta - \varepsilon]$ attains its minimum, and that value is the global minimum on $(0, u_\eta)$. As both summands are positive, so the minimum is positive. Hence $\mathcal{V}_{\delta,k}$ is bounded below by a positive constant. Its sublevel sets $\{\mathcal{V}_{\delta,k} \leq n\}$ are closed subsets of $[\varepsilon_n, u_\eta - \varepsilon_n]$ for some $\varepsilon_n > 0$, therefore compact in $(0, u_\eta)$. We impose the restriction $\delta < \gamma_0$, where γ_0 is defined in (A.14). This assumption will make the lower-end generator ratio strictly negative. Any $k > 0$ works at the upper end, because the inward drift will send the ratio to $-\infty$ as we show below.

We now calculate the geneator of each component in (A.35) using $\mathcal{L}_\eta \psi = \mu_\eta \psi' + \zeta_\eta^2 \psi''/2$. Start with $\psi(q) = q^{-\delta}$:

$$\frac{\mathcal{L}_\eta q^{-\delta}}{q^{-\delta}} = -\delta \frac{\mu_\eta(q)}{q} + \frac{\delta(\delta+1)}{2} (\sigma_\eta^s(q))^2.$$

²⁷The invariant density is identified once diverging scale rules out a non-zero probability current.

²⁸This is the Foster drift (Meyn and Tweedie, 1993, Theorem 3.1).

By definition $\ell_\eta(q) = \mu_\eta(q)/q - (\sigma_\eta^s(q))^2/2$,²⁹ hence $\mu_\eta(q)/q = \ell_\eta(q) + (\sigma_\eta^s(q))^2/2$. Insert this:

$$\frac{\mathcal{L}_\eta q^{-\delta}}{q^{-\delta}} = -\delta \ell_\eta(q) + \frac{\delta^2}{2} (\sigma_\eta^s(q))^2. \quad (\text{A.36})$$

Now for the second function $\psi(q) = (u_\eta - q)^{-k}$, first write $w = u_\eta - q$, and then we obtain

$$\frac{\mathcal{L}_\eta (u_\eta - q)^{-k}}{(u_\eta - q)^{-k}} = \frac{k\mu_\eta(q)}{u_\eta - q} + \frac{k(k+1)\zeta_\eta(q)^2}{2(u_\eta - q)^2}. \quad (\text{A.37})$$

The generator is linear, so

$$\mathcal{L}_\eta \mathcal{V}_{\delta,k} = \mathcal{L}_\eta q^{-\delta} + \mathcal{L}_\eta (u_\eta - q)^{-k}.$$

Write $\mathcal{V}_- = q^{-\delta}$ and $\mathcal{V}_+ = (u_\eta - q)^{-k}$. Then

$$\frac{\mathcal{L}_\eta \mathcal{V}_{\delta,k}}{\mathcal{V}_{\delta,k}} = \frac{\mathcal{L}_\eta \mathcal{V}_-}{\mathcal{V}_-} \cdot \frac{\mathcal{V}_-}{\mathcal{V}_- + \mathcal{V}_+} + \frac{\mathcal{L}_\eta \mathcal{V}_+}{\mathcal{V}_- + \mathcal{V}_+}.$$

Near zero, $\mathcal{V}_+ \rightarrow u_\eta^{-k}$ and $\mathcal{L}_\eta \mathcal{V}_+$ stays bounded, because $(u_\eta - q)^{-k}$ is C^2 on a neighborhood of 0 inside $(0, u_\eta)$. Also $\mathcal{V}_- \rightarrow +\infty$, so $\mathcal{V}_-/(\mathcal{V}_- + \mathcal{V}_+) \rightarrow 1$ and $(\mathcal{L}_\eta \mathcal{V}_+)/(\mathcal{V}_- + \mathcal{V}_+) \rightarrow 0$. Hence

$$\frac{\mathcal{L}_\eta \mathcal{V}_{\delta,k}(q)}{\mathcal{V}_{\delta,k}(q)} \longrightarrow -\delta \ell_0 + \frac{\delta^2}{2} (\bar{\sigma}^s)^2,$$

where we know that $\ell_\eta(q) \rightarrow \ell_0$ from (A.12). Rewrite the limit with $\gamma_0 = 2\ell_0/(\bar{\sigma}^s)^2$:

$$-\delta \ell_0 + \frac{\delta^2}{2} (\bar{\sigma}^s)^2 = \frac{\delta (\bar{\sigma}^s)^2}{2} \left(\delta - \frac{2\ell_0}{(\bar{\sigma}^s)^2} \right) = \frac{\delta (\bar{\sigma}^s)^2}{2} (\delta - \gamma_0).$$

The assumption $0 < \delta < \gamma_0$ makes this limit strictly negative.

Near $q \sim u_\eta$, the roles reverse. $\mathcal{V}_-$ and $\mathcal{L}_\eta \mathcal{V}_-$ stay bounded, while $\mathcal{V}_+ \rightarrow +\infty$. The limits in (A.28) and (A.29), $\mu_\eta \rightarrow \mu_u < 0$ and $\zeta_\eta \sim u_\eta \zeta_u(u_\eta - q)$, give

$$\frac{\mathcal{L}_\eta \mathcal{V}_{\delta,k}(q)}{\mathcal{V}_{\delta,k}(q)} \sim \frac{k\mu_u}{u_\eta - q} + \frac{k(k+1)}{2} u_\eta^2 \zeta_u^2.$$

The first term tends to $-\infty$ and dominates the finite second term. Thus the ratio for the full $\mathcal{V}_{\delta,k}$ tends to minus infinity at u_η .

Choose neighborhoods of 0 and of u_η on which the ratio is at most $-c_\gamma$ for some $c_\gamma > 0$.

²⁹Recall that $\ell_\eta(q)$ is the drift of $\log q$, defined in (A.11).

Then $\mathcal{L}_\eta \mathcal{V}_{\delta,k} \leq -c_V \mathcal{V}_{\delta,k}$ there. On the remaining compact subinterval both $\mathcal{V}_{\delta,k}$ and $\mathcal{L}_\eta \mathcal{V}_{\delta,k}$ are continuous, so $\mathcal{L}_\eta \mathcal{V}_{\delta,k} + c_V \mathcal{V}_{\delta,k}$ is bounded above by a finite constant K_V . Combining the two regions yields the Foster inequality (Meyn and Tweedie, 1993)

$$\mathcal{L}_\eta \mathcal{V}_{\delta,k}(q) \leq -c_V \mathcal{V}_{\delta,k}(q) + K_V. \quad (\text{A.38})$$

These constants, c_V and K_V can depend on primitive parameters of the process. The additive K_V term absorbs the compact middle of the interval, where the process need not be pulled. The negative multiple of $\mathcal{V}_{\delta,k}$ is the pull in the two tails.

The process has a unique strong solution only up to the first exit from $(0, u_\eta)$. Localization turns the Foster inequality into a bound that prevents that exit. Because $\mathcal{V}_{\delta,k}$ is continuous and proper, the set $\{\mathcal{V}_{\delta,k} \geq n\}$ lies in a pair of neighborhoods of 0 and u_η and does not meet a fixed interior compact. Let τ_n be the first time $\mathcal{V}_{\delta,k}(q_t)$ equals a positive integer n , and fix a horizon $T < \infty$. On $[0, \tau_n]$ the path stays in a compact subset of $(0, u_\eta)$, where μ_η and ζ_η are Lipschitz and $\mathcal{V}_{\delta,k}$ is C^2 and bounded. Apply (S.1.1) to $\psi = \mathcal{V}_{\delta,k}$:

$$d\mathcal{V}_{\delta,k}(q_t) = \mathcal{L}_\eta \mathcal{V}_{\delta,k}(q_t) dt + \zeta_\eta(q_t) \mathcal{V}'_{\delta,k}(q_t) dZ_t.$$

Integrate from date 0 to the stopped time $T \wedge \tau_n$:

$$\mathcal{V}_{\delta,k}(q_{T \wedge \tau_n}) = \mathcal{V}_{\delta,k}(q) + \int_0^{T \wedge \tau_n} \mathcal{L}_\eta \mathcal{V}_{\delta,k}(q_s) ds + \int_0^{T \wedge \tau_n} \zeta_\eta(q_s) \mathcal{V}'_{\delta,k}(q_s) dZ_s.$$

On $[0, \tau_n]$ the state remains in a compact subset of $(0, u_\eta)$. There ζ_η and $\mathcal{V}'_{\delta,k}$ are continuous, hence bounded. The integrand of the stochastic integral is therefore square-integrable on $[0, T]$, so that integral is a true martingale and has mean zero under $\mathbb{E}_q$. Taking expectations leaves only the generator integral:

$$\mathbb{E}_q[\mathcal{V}_{\delta,k}(q_{T \wedge \tau_n})] = \mathcal{V}_{\delta,k}(q) + \mathbb{E}_q \left[\int_0^{T \wedge \tau_n} \mathcal{L}_\eta \mathcal{V}_{\delta,k}(q_s) ds \right].$$

The Foster inequality (A.38) implies $\mathcal{L}_\eta \mathcal{V}_{\delta,k}(q) \leq K_V$ at every interior state, since $\mathcal{V}_{\delta,k}(q) \geq 0$. The integrand is therefore at most the constant K_V , and the length of the integration interval is at most T :

$$\int_0^{T \wedge \tau_n} \mathcal{L}_\eta \mathcal{V}_{\delta,k}(q_s) ds \leq \int_0^{T \wedge \tau_n} K_V ds = K_V(T \wedge \tau_n) \leq K_V T,$$

implying

$$\mathbb{E}_q[\mathcal{V}_{\delta,k}(q_{T \wedge \tau_n})] \leq \mathcal{V}_{\delta,k}(q) + K_V T.$$

Interestingly, on the event $\{\tau_n \leq T\}$ one has $\mathcal{V}_{\delta,k}(q_{\tau_n}) = n$, so

$$n \mathbb{P}_q(\tau_n \leq T) \leq \mathbb{E}_q[\mathcal{V}_{\delta,k}(q_{T \wedge \tau_n})] \leq \mathcal{V}_{\delta,k}(q) + K_V T.$$

Thus

$$\mathbb{P}_q(\tau_n \leq T) \leq \frac{\mathcal{V}_{\delta,k}(q) + K_V T}{n}.$$

With $n \rightarrow \infty$, the right-hand side tends to 0, and $\mathbb{P}_q(\tau_n \leq T) \rightarrow 0$ for every finite T . The explosion time $\tau_\infty = \lim_n \tau_n$ is thus infinite almost surely. But $\mathcal{V}_{\delta,k}$ diverges at both endpoints, so τ_∞ is also the first exit from $(0, u_\eta)$, implying that exit cannot occur in finite time. The local solution therefore exists for all $t \geq 0$, and the pathwise uniqueness already obtained up to exit extends to every finite horizon.³⁰

We can also obtain the expected Lyapunov bound. Apply (S.1.1) to the product $e^{c_V t} \mathcal{V}_{\delta,k}(q_t)$. The ordinary product rule and the generator produce the drift

$$e^{c_V t} (c_V \mathcal{V}_{\delta,k}(q_t) + \mathcal{L}_\eta \mathcal{V}_{\delta,k}(q_t)).$$

The Foster inequality (A.38) is exactly $c_V \mathcal{V}_{\delta,k} + \mathcal{L}_\eta \mathcal{V}_{\delta,k} \leq K_V$. Stopping at $t \wedge \tau_n$ again makes the stochastic integral a martingale, so

$$\mathbb{E}_q[e^{c_V(t \wedge \tau_n)} \mathcal{V}_{\delta,k}(q_{t \wedge \tau_n})] \leq \mathcal{V}_{\delta,k}(q) + K_V \mathbb{E}_q \left[\int_0^{t \wedge \tau_n} e^{c_V s} ds \right] \leq \mathcal{V}_{\delta,k}(q) + \frac{K_V}{c_V} (e^{c_V t} - 1).$$

Send $n \rightarrow \infty$. Almost sure non-explosion gives $t \wedge \tau_n \rightarrow t$ and $q_{t \wedge \tau_n} \rightarrow q_t$. Fatou's lemma passes the expectation to the left-hand side and yields $\mathbb{E}_q[e^{c_V t} \mathcal{V}_{\delta,k}(q_t)] \leq \mathcal{V}_{\delta,k}(q) + (K_V/c_V)(e^{c_V t} - 1)$. Multiply by $e^{-c_V t}$:

$$\mathbb{E}_q \mathcal{V}_{\delta,k}(q_t) \leq e^{-c_V t} \mathcal{V}_{\delta,k}(q) + \frac{K_V}{c_V} (1 - e^{-c_V t}). \quad (\text{A.39})$$

The expected Lyapunov value therefore stays finite at every finite time, and it is bounded uniformly in t by $\mathcal{V}_{\delta,k}(q) + K_V/c_V$.

We now turn to convergence of the distribution of q again. While Lyapunov control pulls the process back from the tails on average, that pullback alone does not give nearby interior starts overlapping successor laws, so the distribution might not forget its initial condition. Harris recurrence (Hairer and Mattingly, 2011; Hairer, 2018) supplies the missing ingredient: besides the Foster bound, it requires that overlap.

³⁰Note that this result is consistent with the scale classification above, where infinite scale distance already makes both endpoints inaccessible from the interior.

A set C is small, in the sense used below, if the one-step kernel from every point of C is bounded below by a common positive multiple of a single probability measure ν :³¹

$$\mathbb{P}_q(q_\Delta \in A) \geq \varepsilon \nu(A) \quad \text{for every } q \in C \text{ and every Borel } A. \quad (\text{A.40})$$

The interpretation of (A.40) is straightforward. After one sampling step (Δ), two trajectories that both sit in C can be coupled onto the ν -component with probability ε , and from that instant they can be driven by the same process, ensuring that each return to C is a chance to forget the exact interior location. Together with the Foster inequality that makes such returns occur with uniformly integrable waiting times, the memory of the initial state is erased.

That guarantees uniqueness of the invariant probability and convergence in total variation. Without the overlap construction, Lyapunov pullback alone could still permit two interior basins that never mix.

The candidate small sets C are Lyapunov sublevels $\{\mathcal{V}_{\delta,k} \leq R\}$. Properness of $\mathcal{V}_{\delta,k}$ makes every such sublevel a compact subset of $(0, u_\eta)$. Fix one sublevel C and enclose it in a strictly larger compact interval $I \subset (0, u_\eta)$. On I , the loading ζ_η is continuous and bounded away from zero, because $\sigma_\eta^s \geq \sigma > 0$ and q stays away from both endpoints. Note that the larger interval I is used only as a room in which paths may wiggle.

Now we define killing. A path is killed at the first exit from I (or from its image I_x below), when we discard the process from that time onward and no longer record its position. The killed transition density at time Δ is the sub-probability density of being at an interior point without having left the interval: it equals the ordinary density times the indicator that the path stayed inside throughout $[0, \Delta]$. Its integral over the interval gives the probability of no exit, which is strictly less than one.³²

Supplementary Appendix S.3 passes to the Lamperti coordinate $x(q) = \int_{q_0}^q \zeta_\eta^{-1}$ on the interval I just fixed, removes the state dependence of the diffusion coefficient, and identifies the killed physical path measure with the killed Brownian measure reweighted by the exponential martingale $\mathcal{E}_\Delta$. It produces a constant $\varepsilon_\mathcal{E} > 0$, which depends only on Δ , C and I , such that $\mathcal{E}_\Delta \geq \varepsilon_\mathcal{E}$ on surviving paths with endpoints in $x(C)$. The remainder of the argument stays on the q -axis and turns that bound into another constant ε_C defined below.

Write $p^\circ(\Delta, x, y)$ for the killed Brownian transition density on I_x : the density of occupying y at date Δ and of never having left I_x on $[0, \Delta]$. It factors as the free Gaussian times the

³¹The inequality (A.40) does not imply that every start in C has the same successor law. It says that those laws share a common lump of mass ε , distributed exactly as ν , with leftover mass $1 - \varepsilon$ that may still depend on the start.

³²Note that the true process is not killed. It may leave I and return.

probability that the Brownian bridge from x to y in time Δ stays inside I_x :

$$p^\circ(\Delta, x, y) = (2\pi\Delta)^{-1/2} \exp\left(-(y-x)^2/(2\Delta)\right) \cdot \mathbb{P}\left(\text{bridge from } x \text{ to } y \text{ stays in } I_x\right). \quad (\text{A.41})$$

On the compact product $x(C) \times x(C)$ the Gaussian is continuous and positive. The distance $|y-x|$ cannot exceed the diameter of $x(C)$, so the Gaussian is at least

$$\varepsilon_G := (2\pi\Delta)^{-1/2} \exp\left(-\text{diam}(x(C))^2/(2\Delta)\right) > 0.$$

The second factor in (A.41) is the probability that the path stays in I_x , conditional on the endpoints (x, y) . That conditional process can be described as the Brownian bridge from x to y in time Δ , which is free Brownian motion on $[0, \Delta]$, restricted to the event $\{W_\Delta = y\}$ when $W_0 = x$. The standard decomposition separates a deterministic segment from a mean-zero fluctuation. Let W be a free Brownian motion starting at 0, and set $\sigma_s = W_s - (s/\Delta)W_\Delta$. Then $\sigma_0 = \sigma_\Delta = 0$, and the bridge is

$$X_s^{x \rightarrow y} = x + \frac{s}{\Delta}(y-x) + \sigma_s, \quad s \in [0, \Delta].$$

The first two terms trace the straight segment from x to y . σ_s is a standard Brownian bridge from 0 to 0. Its law does not depend on the endpoints (x, y) .

The interval I_x is one-dimensional. Write it as a compact interval $[\alpha, \beta]$. The set $x(C)$ ³³ lies in a strictly smaller compact sub-interval, because C was enclosed in the interior of I . Hence there is a gap $d > 0$ such that every point of $x(C)$ lies in $[\alpha + d, \beta - d]$. The segment joining any $x, y \in x(C)$ is the ordinary interval between them, so it remains in $[\alpha + d, \beta - d]$ as well. If $\sup_{s \leq \Delta} |\sigma_s| < d$, then

$$\alpha < x + \frac{s}{\Delta}(y-x) + \sigma_s < \beta \quad \text{for every } s \in [0, \Delta],$$

so the whole bridge stays in I_x . Therefore

$$\mathbb{P}\left(X^{x \rightarrow y} \text{ stays in } I_x\right) \geq \mathbb{P}\left(\sup_{s \leq \Delta} |\sigma_s| < d\right).$$

The right-hand side does not depend on (x, y) . It is positive because a continuous process pinned at 0 at both ends remains in $(-d, d)$ on a compact time interval with positive probability. Call that probability $\varepsilon_{\text{br}} > 0$. The same d and the same σ work for every pair in $x(C) \times x(C)$,

³³Recall that $x(q)$ is the Lamperti coordinate. $x(C)$ is the set of all x -values of q in C .

so the lower bound is already uniform. Hence

$$p^\circ(\Delta, x, y) \geq \varepsilon_G \varepsilon_{\text{br}} =: \varepsilon^\circ > 0 \quad (\text{A.42})$$

for all $x, y \in x(C)$.

As mentioned above, Supplementary Appendix S.3 changes pure Brownian motion into the Lamperti process $x(q) = \int_{q_0}^q \zeta_\eta^{-1} dq$ using Girsanov's theorem: basically, each surviving path is multiplied by the exponential martingale $\mathcal{E}_\Delta$. Note that the lower bound in (A.42) is for the killed Brownian density $p^\circ(\Delta, x, y)$ of $dx = dW$ until exit. We convert p° into the killed Lamperti density $p^{\text{kill}}(\Delta, x, y)$ of $dx = \mathfrak{b}(x) dt + dW$ until exit, which is the density for the physical x -process, and characterize its lower bound.

On surviving paths, $dQ_{\text{phys}} = \mathcal{E}_\Delta dP$. Therefore the sub-probability of occupying y at date Δ without having left I_x is

$$p^{\text{kill}}(\Delta, x, y) dy = \mathbb{E}_P \left[1_{\{x_\Delta \in dy, \text{no exit}\}} \mathcal{E}_\Delta \right] = \mathbb{E}_P \left[\mathcal{E}_\Delta \mid x_\Delta = y, \text{no exit} \right] p^\circ(\Delta, x, y) dy.$$

The conditional expectation is an average of $\mathcal{E}_\Delta$ over Brownian bridges from x to y that stay in I_x . On every such path $\mathcal{E}_\Delta \geq \varepsilon_\mathcal{E}$ as shown in Supplementary Appendix S.3, so the average is at least $\varepsilon_\mathcal{E}$. Hence

$$p^{\text{kill}}(\Delta, x, y) = \mathbb{E}_P \left[\mathcal{E}_\Delta \mid x_\Delta = y, \text{no exit} \right] p^\circ(\Delta, x, y) \geq \varepsilon_\mathcal{E} p^\circ(\Delta, x, y) \geq \varepsilon_\mathcal{E} \varepsilon^\circ \quad (\text{A.43})$$

uniformly for $x, y \in x(C)$. Because the true Lamperti process on x -axis is not killed, any path that leaves I_x and returns is counted by the true density and ignored by the killed density, so the true x -density is at least p^{kill} .

Now that we have the lower bound in (A.43) for the true and killed Lamperti density, we easily transform it to the q -coordinate. Transforming the density from x to q multiplies by the Jacobian $x'(q) = 1/\zeta_\eta(q)$. If p_x is a density in the x -coordinate, the corresponding q -density is $p_q(q_0, q) = p_x(x(q_0), x(q))/\zeta_\eta(q)$. On C the loading ζ_η is continuous and positive, hence bounded above by some $\zeta_C < \infty$, and

$$\frac{1}{\zeta_\eta(q)} \geq \frac{1}{\zeta_C} > 0.$$

Therefore the true q -transition density $p_q(\Delta, q_0, q)$ obeys, for every $q_0, q \in C$,

$$p_q(\Delta, q_0, q) \geq \frac{p^{\text{kill}}(\Delta, x(q_0), x(q))}{\zeta_\eta(q)} \geq \frac{\varepsilon_\mathcal{E} \varepsilon^\circ}{\zeta_C} =: \varepsilon_C > 0. \quad (\text{A.44})$$

The lower bound ε_C is the product of the Girsanov lower bound $\varepsilon_{\mathcal{E}}$, the killed Brownian lower bound ε° and the reciprocal of the maximal loading on C . It depends on Δ , C and I , but not on the starting point inside C . Integrating (A.44) over $A \cap C$ gives

$$\mathbb{P}_q(q_\Delta \in A) \geq \varepsilon_C \text{Leb}(A \cap C), \quad (\text{A.45})$$

or equivalently $\mathbb{P}_q(q_\Delta \in A) \geq \varepsilon \text{Leb}(A)$ with $\varepsilon = \varepsilon_C \text{Leb}(C)$.

The sampled Lyapunov bound follows from (A.39) at time Δ :

$$\mathbb{E}_q \mathcal{V}_{\delta,k}(q_\Delta) \leq e^{-c\nu\Delta} \mathcal{V}_{\delta,k}(q) + \frac{K_{\mathcal{V}}}{c_{\mathcal{V}}} (1 - e^{-c\nu\Delta}).$$

The contraction factor $e^{-c\nu\Delta}$ is strictly smaller than one, and the additive constant is finite. Combined with (A.45), it is Assumptions 1–2 of Hairer and Mattingly (2011), implying that the process has a unique invariant probability and converges to it in a weighted total-variation norm. Finally, the mixing condition (A.45) implies ergodicity, so the ergodic theorem gives almost-sure time averages for the integrable observables just considered (Hairer, 2018). $\square$

A.5 The effective lower bound

Proof of Proposition 7. Differentiating the Taylor prescription gives $i'(q) = \phi_y + \phi_\pi \beta \nu q^{\nu-1} > 0$ on $(0, u_\eta)$. Hence $i(0+) < i(u_\eta)$, and a floor strictly between them is attained at exactly one $q_{\text{elb}} \in (0, u_\eta)$. For $q < q_{\text{elb}}$, $i(q) < i^{\text{elb}}$, so $H_B(q) > H(q)$ and the floor i^{elb} binds. For $q > q_{\text{elb}}$, $H_B(q) < H(q)$. At q_{elb} , $i(q_{\text{elb}}) = i^{\text{elb}}$ is $H(q_{\text{elb}}) = H_B(q_{\text{elb}})$, so H_{elb} and $F_{\eta, \text{elb}}$ are continuous.

On $[q_{\text{elb}}, u_\eta]$, $F_{\eta, \text{elb}} = F_\eta$. Lemma 1 gives that this function is strictly decreasing, positive on $[q_{\text{elb}}, u_\eta]$, and zero at u_η . On $(0, q_{\text{elb}})$, $H_{\text{elb}}(q) = H_B(q)$ and $H'_B(q) = -\pi'(q) = -\beta \nu q^{\nu-1}$. Differentiating (3.30) therefore produces (3.32). The first term is negative because $\kappa > 0$, $\beta > 0$ and $m > \nu$. The second term is also negative because $0 < \nu < (\varepsilon - 1)/2$ from (3.31). Therefore, F_B is strictly decreasing on $(0, q_{\text{elb}})$. Since $H_B > H$ there, $F_B < F_\eta$ on $(0, q_{\text{elb}})$, and the two functions agree at q_{elb} . A continuous function that decreases strictly on each of $(0, q_{\text{elb}}]$ and $[q_{\text{elb}}, u_\eta]$ decreases strictly on $(0, u_\eta]$. Its only zero is u_η , so $F_{\eta, \text{elb}} > 0$ on $(0, u_\eta)$. The volatility root (3.5) with $F_{\eta, \text{elb}} > 0$ in place of F_η is therefore positive on the interior.

The resulting drift and diffusion agree with the unconstrained coefficients on (q_{elb}, u_η) . At the single interior kink both one-sided derivatives are finite, so the coefficients are locally Lipschitz on $(0, u_\eta)$. Near zero the floor binds, and $H_{\text{elb}}(0+) = i^{\text{elb}} - r^n - \underline{\pi}$ is strictly larger than $H(0+) = -\Phi$. Effectively, the real rate r_t gets closer to the natural rate r^n than without

the lower bound.

In (A.12), this replaces Φ by the smaller shortfall $r^n + \underline{\pi} - i^{\text{elb}}$. The maintained restriction $\nu^2 \Phi < \omega$ keeps the limiting drift strictly positive, and that drift points away from the origin. The upper endpoint is untouched. The conclusions of Proposition 3 therefore apply to the censored state equation, and the strictly positive invariant density assigns positive probability to both $(0, q_{\text{elb}})$ and (q_{elb}, u_η) .

□

B Stationary equilibrium properties

This section proves Proposition 4 and Proposition 5. First, Proposition 4 records the signs of the stationary observables and their comovements.

Proof of Proposition 4. From Lemma 1, $0 < q < u_\eta < 1$, $\pi(q) < 0$ and $H(q) < 0$. From (3.3),

$$\begin{aligned} \pi'(q) &= \beta \nu q^{\nu-1} > 0, & H'(q) &= \phi_y + (\phi_\pi - 1)\pi'(q) > 0, \\ i'(q) &= \phi_y + \phi_\pi \pi'(q) > 0, & (\sigma_\eta^s)'(q) &= \frac{F'_\eta(q)}{\sigma + (1 + \nu)\sigma_\eta^s(q)} < 0. \end{aligned} \quad (\text{B.1})$$

Thus the limits at zero and u_η prove (3.12) and (3.13), with $H(0+) = -\Phi$, $\sigma_\eta^s(0+) = \bar{\sigma}^s$ and $\sigma_\eta^s(u_\eta) = 0$. Then $L = L^n q$, $\text{mc} = \alpha q^m$ and $D/(pC) = 1 - \alpha q^m$ prove (3.14).

The gap $q - 1$ and inflation $\pi(q)$ are strictly increasing in q , while excess volatility $\sigma_\eta^s(q)$ is strictly decreasing. Each function is bounded on $(0, u_\eta)$. Under a law with strictly positive density, the covariance of two strictly increasing functions of the state is strictly positive, and the covariance of a strictly increasing function and a strictly decreasing function is strictly negative, proving (3.15). □

Proposition 5 then converts the invariant density into mean formulas.

Proof of Proposition 5. Proposition 3 proves $0 < N_\eta < \infty$. For every $\theta \geq 0$, q^θ is bounded on the state interval, so (3.16) is finite and equals $\mathbb{E}_\star[q^\theta]$. Integration of $\tilde{Y} = q - 1$ and (3.3) gives (3.17). Full interior support and the strict pointwise bounds imply strictly interior means. Under stationarity, if q_0 is drawn from the invariant law, then q_T has the same law for every T . Hence,

$$\mathbb{E}_\star[\log q_T] = \mathbb{E}_\star[\log q_0].$$

The drift of the log ratio is the bounded function

$$\ell_\eta(q) = H(q) + \sigma \sigma_\eta^s(q) + \frac{1}{2} \left(\sigma_\eta^s(q) \right)^2,$$

and its diffusion loading $\sigma_\eta^s(q)$ is bounded. Thus,

$$\log q_T - \log q_0 = \int_0^T \ell_\eta(q_t) dt + \int_0^T \sigma_\eta^s(q_t) dZ_t. \quad (\text{B.2})$$

The stochastic integral is square-integrable, with second moment bounded above by $T(\bar{\sigma}^s)^2$, and therefore has zero expectation. Taking expectations in (B.2) yields

$$\mathbb{E}_\star[\log q_T] - \mathbb{E}_\star[\log q_0] = \mathbb{E}_\star \left[\int_0^T \ell_\eta(q_t) dt \right].$$

The left-hand side vanishes by invariance of the law of q_t . Since the integrand is bounded, Fubini's theorem and stationarity imply

$$0 = T \mathbb{E}_\star[\ell_\eta(q)],$$

so that $\mathbb{E}_\star[\ell_\eta(q)] = 0$.³⁴ Substituting the definition of H establishes (3.18).

For $\pi(q)$, the drift in (A.10) is bounded, as is the diffusion loading $\beta \nu q^\nu \sigma_\eta^s(q)$. Consequently, the stochastic integral in the integrated inflation equation is also a square-integrable martingale on every finite time horizon. Since $\pi(q)$ itself is bounded, taking expectations of the integrated equation under invariant initialization gives

$$0 = T \mathbb{E}_\star \left[\rho \pi(q) - \kappa(q^m - 1) - \frac{\varepsilon - 1}{2} \pi(q)^2 \right],$$

which establishes (3.19).

Equations (3.20), (3.21), and (3.22) are rearrangements of these identities. First, since

$$H(q) = \phi_y(q - 1) + (\phi_\pi - 1)\pi(q) \quad \text{and} \quad r(q) = r^n + H(q),$$

we have

$$\mathbb{E}_\star[H(q)] = \phi_y \mathbb{E}_\star[\tilde{Y}] + (\phi_\pi - 1) \mathbb{E}_\star[\pi].$$

³⁴Note that this condition is an invariant-average restriction on the drift $\ell_\eta(q)$, not a pointwise condition requiring $\ell_\eta(q) = 0$ at every state.

Equation (3.18) is simply the condition $\mathbb{E}_\star[\ell_\eta(q)] = 0$, written as

$$\mathbb{E}_\star[H(q)] + \sigma \mathbb{E}_\star[\sigma^s] + \frac{1}{2} \mathbb{E}_\star[(\sigma^s)^2] = 0.$$

Since $\phi_y > 0$, solving for the mean output gap yields (3.20). The same condition also implies

$$\mathbb{E}_\star[r^n - r(q)] = -\mathbb{E}_\star[H(q)],$$

leading to (3.22). With $\sigma_\eta^s(q) > 0$ in the interior, while the invariant density is strictly positive there, both $\mathbb{E}_\star[\sigma^s]$ and $\mathbb{E}_\star[(\sigma^s)^2]$ are strictly positive.

Equation (3.19) is already linear in $\mathbb{E}_\star[\pi]$. Dividing by $\rho > 0$ and isolating this expectation yields (3.21). Finally, because $0 < \nu < 1$, the function $q \mapsto q^\nu$ is strictly concave. Since the invariant law is non-degenerate, strict Jensen's inequality gives

$$M_{\eta,\nu} < M_{\eta,1}^\nu,$$

establishing (3.23). □

Supplementary Appendix

Proposition 3 states that the output ratio q_t in (3.6) never leaves $(0, u_\eta)$, features the invariant density (3.10), and converges to the stationary distribution from every interior point q . While Appendix A.4 proves the statement, three blocks of the proof rely on diffusion calculations that are not special to the New Keynesian coefficients, and those calculations are separately explained here for clarity.

In specific, Supplementary Appendix S.1 first derives the scale function, the speed density, and the stationary Kolmogorov forward equation. The proof in Appendix A.4 then evaluates the resulting integrals at $q \simeq 0$ and at $q \simeq u_\eta$, and uses the range of the scale function to pin down the probability distribution. Supplementary Appendix S.2 shows that the drift and diffusion of (3.6) extend Lipschitz-continuously to the closed interval, which is the input to the strong-existence part of the proof. Supplementary Appendix S.3 constructs the unit-diffusion coordinate and the Girsanov weight used in Appendix A.4 minorizes transition probabilities on a compact interior set. Each section explains the step of Appendix A.4 that consumes it.

S.1 Scale, speed, and boundary tests

The stationary density in (3.10) is the normalized solution of the forward equation for (3.6). The boundary tests in Appendix A.4 are integrals of the scale density and of the speed density of the same diffusion. In this section, we derive both densities, and the Kolmogorov forward equation, from Itô's formula.³⁵

The state follows (3.6) up to exit from $(0, u_\eta)$. For a twice continuously differentiable test function ψ , Itô's formula and the generator defined after (3.6) give

$$d\psi(q_t) = \mathcal{L}_\eta \psi(q_t) dt + \zeta_\eta(q_t) \psi'(q_t) dZ_t. \quad (\text{S.1.1})$$

The generator $\mathcal{L}_\eta \psi = \mu_\eta \psi' + \zeta_\eta^2 \psi''/2$ is the drift of $\psi(q_t)$. A scale function, $\mathfrak{s}_\eta$, is a *strictly increasing* C^2 function that satisfies

$$\mathcal{L}_\eta \mathfrak{s}_\eta = 0.$$

Strict monotonicity makes $\mathfrak{s}_\eta(q)$ a possible coordinate change on the state interval (q -axis): distinct output ratios receive distinct scale values, so the q -axis can merely be relabeled.³⁶

³⁵The actual numerical values of the endpoint integrals are computed in the proof in Appendix A.4, not here.

³⁶Its name, scale, means this relabeling.

Substituting $\psi = \mathfrak{s}_\eta$ into (S.1.1) cancels the drift,

$$d\mathfrak{s}_\eta(q_t) = \zeta_\eta(q_t)\mathfrak{s}'_\eta(q_t)dZ_t,$$

so $\mathfrak{s}_\eta(q_t)$ is a local martingale. In the original coordinate the state still has drift μ_η . In the scale coordinate the same path is driftless. This driftless property allows us to calculate the hitting probabilities derived next, which are given by ratios of scale distances. Appendix A.4 specializes those ratios to the endpoints of $(0, u_\eta)$.

Fix generic barriers $0 < a < q < b < u_\eta$, where a and b are state levels. Write $\mathfrak{h}_a$ for the first time the state q_t equals a starting from q , and $\mathfrak{h}_{a,b} = \mathfrak{h}_a \wedge \mathfrak{h}_b$. On the compact interval $[a, b]$ the processes $\mathfrak{s}_\eta(q_t)$ and $\zeta_\eta(q_t)\mathfrak{s}'_\eta(q_t)$ are bounded up to $\mathfrak{h}_{a,b}$, so the stopped scale process is a true martingale. Optional sampling at $\mathfrak{h}_{a,b}$ yields

$$\mathfrak{s}_\eta(q) = \mathfrak{s}_\eta(a)\mathbb{P}_q(\mathfrak{h}_a < \mathfrak{h}_b) + \mathfrak{s}_\eta(b)\mathbb{P}_q(\mathfrak{h}_b < \mathfrak{h}_a).$$

The two probabilities sum to one, and therefore the hitting probability is given by a ratio of scale distances:

$$\mathbb{P}_q(\mathfrak{h}_a < \mathfrak{h}_b) = \frac{\mathfrak{s}_\eta(b) - \mathfrak{s}_\eta(q)}{\mathfrak{s}_\eta(b) - \mathfrak{s}_\eta(a)}. \quad (\text{S.1.2})$$

If $\mathfrak{s}_\eta(a) \rightarrow -\infty$ as $a \downarrow 0$, then for each fixed interior b the probability of hitting a before b vanishes. If $\mathfrak{s}_\eta(b) \rightarrow +\infty$ as $b \uparrow u_\eta$, the probability of hitting b before a fixed interior a likewise vanishes. Thus, infinite scale distance from an interior point to an endpoint is necessary and sufficient for that endpoint to be unreachable in finite time.

The ratios in (S.1.2) are evaluated once an explicit scale function is available. The equation $\mathcal{L}_\eta \mathfrak{s}_\eta = 0$ is the first-order ODE $\mu_\eta \mathfrak{s}'_\eta + (\zeta_\eta^2/2)\mathfrak{s}''_\eta = 0$. Wherever $\mathfrak{s}'_\eta \neq 0$, it rearranges as

$$\frac{\mathfrak{s}''_\eta(q)}{\mathfrak{s}'_\eta(q)} = -\frac{2\mu_\eta(q)}{\zeta_\eta(q)^2}.$$

The left-hand side is $\partial_q \log \mathfrak{s}'_\eta$. Integrating from the interior reference $q_\circ$ in (3.9) and imposing $\mathfrak{s}'_\eta(q_\circ) = 1$ gives the unique positive scale density $\mathfrak{s}'_\eta = e^{-\mathcal{R}_\eta}$. The associated scale function, normalized by $\mathfrak{s}_\eta(q_\circ) = 0$, is thus given by

$$\begin{aligned} \mathfrak{s}'_\eta(q) &= \exp \left[-\int_{q_\circ}^q \frac{2\mu_\eta(\tilde{q})}{\zeta_\eta(\tilde{q})^2} d\tilde{q} \right], \\ \mathfrak{s}_\eta(q) &= \int_{q_\circ}^q \mathfrak{s}'_\eta(\tilde{q}) d\tilde{q}. \end{aligned} \quad (\text{S.1.3})$$

Any other strictly increasing solution differs by a positive affine transformation, which cancels in (S.1.2). With this normalization, $\mathfrak{s}_\eta(a) \rightarrow -\infty$ is exactly $\int_0^{q_0} \mathfrak{s}'_\eta = \infty$, and $\mathfrak{s}_\eta(b) \rightarrow +\infty$ is $\int_{q_0}^{u_\eta} \mathfrak{s}'_\eta = \infty$. We verify these two conditions in Appendix A.4.

Note that the scale function does not record how long the path spends near each point. Hitting concerns only the order of visits. In contrast, in order to obtain an invariant probability, we need to consider occupation: even if an endpoint is never reached, for example, the process may linger nearby and pile up mass, or pass quickly and leave negligible mass.³⁷

That weight is the speed measure, with Lebesgue density $\mathfrak{m}_\eta$. For a Borel set $A \subset (0, u_\eta)$, the integral $\int_A \mathfrak{m}_\eta(q) dq$ is the speed mass of A , and it is proportional to the expected calendar time the path spends in A . To get the formula for $\mathfrak{m}_\eta$, we start from $\mathfrak{s}''_\eta/\mathfrak{s}'_\eta = -2\mu_\eta/\zeta_\eta^2$. Then

$$\partial_q \left(\frac{1}{\mathfrak{s}'_\eta} \right) = \frac{2\mu_\eta}{\zeta_\eta^2 \mathfrak{s}'_\eta}.$$

For any C^2 test function ψ ,

$$\partial_q \left(\frac{\psi'}{\mathfrak{s}'_\eta} \right) = \frac{1}{\mathfrak{s}'_\eta} \left(\psi'' + \frac{2\mu_\eta}{\zeta_\eta^2} \psi' \right).$$

Multiply by $\zeta_\eta^2 \mathfrak{s}'_\eta/2$:

$$\frac{\zeta_\eta^2 \mathfrak{s}'_\eta}{2} \partial_q \left(\frac{\psi'}{\mathfrak{s}'_\eta} \right) = \mu_\eta \psi' + \frac{1}{2} \zeta_\eta^2 \psi'' = \mathcal{L}_\eta \psi.$$

Define speed as

$$\mathfrak{m}_\eta(q) = \frac{2}{\zeta_\eta(q)^2 \mathfrak{s}'_\eta(q)}, \quad (\text{S.1.4})$$

which leads to:

$$\mathcal{L}_\eta \psi = \frac{1}{\mathfrak{m}_\eta} \partial_q \left(\frac{\psi'}{\mathfrak{s}'_\eta} \right). \quad (\text{S.1.5})$$

This is the scale-speed form of the generator. We can write a similar identity in the coordinate $y = \mathfrak{s}_\eta(q)$ instead of q . Set $\psi(q) = g(y)$ and $Y_t = \mathfrak{s}_\eta(q_t)$. Then $\psi' = g' \mathfrak{s}'_\eta$, so $\psi'/\mathfrak{s}'_\eta = g'$ and $\partial_q(\psi'/\mathfrak{s}'_\eta) = \mathfrak{s}'_\eta g''$. Therefore

$$\mathcal{L}_\eta \psi = \frac{\mathfrak{s}'_\eta}{\mathfrak{m}_\eta} g''.$$

The coefficient is identical to the Itô generator of Y . Indeed $dY_t = \zeta_\eta(q_t) \mathfrak{s}'_\eta(q_t) dZ_t$ and (S.1.4)

³⁷Appendix A.4 uses this occupation weight to decide whether a path that starts arbitrarily close to an endpoint reaches an interior point in finite expected time.

give

$$\frac{\mathfrak{s}'_\eta}{\mathfrak{m}_\eta} = \frac{1}{2} \left(\zeta_\eta \mathfrak{s}'_\eta \right)^2,$$

so

$$\mathcal{L}^Y g(y) = \frac{1}{2} \left(\zeta_\eta(q) \mathfrak{s}'_\eta(q) \right)^2 g''(y) = \frac{\mathfrak{s}'_\eta(q)}{\mathfrak{m}_\eta(q)} g''(y).$$

Define m^Y such that $\mathcal{L}^Y g = g''/\mathfrak{m}^Y$:

$$\frac{1}{\mathfrak{m}^Y(y)} = \frac{\mathfrak{s}'_\eta(q)}{\mathfrak{m}_\eta(q)} = \frac{1}{2} \left(\zeta_\eta(q) \mathfrak{s}'_\eta(q) \right)^2.$$

Inverting produces

$$\mathfrak{m}^Y(y) = \frac{\mathfrak{m}_\eta(q)}{\mathfrak{s}'_\eta(q)} = \frac{2}{\left(\zeta_\eta(q) \mathfrak{s}'_\eta(q) \right)^2}.$$

With this, if a positive measure has Lebesgue density $\mathfrak{m}_\eta$ on the q -axis and $y = \mathfrak{s}_\eta(q)$, then we have $dy = \mathfrak{s}'_\eta(q) dq$ and

$$\int_A \mathfrak{m}_\eta(q) dq = \int_{\mathfrak{s}_\eta(A)} \mathfrak{m}^Y(y) dy \tag{S.1.6}$$

for $A \subset (0, u_\eta)$. Note that even if written in different spaces, both sides of (S.1.6) are speed mass.³⁸

Recall that scale removes drift. Speed, by contrast, restores the clock that this relabeling distorts: where $\zeta_\eta \mathfrak{s}'_\eta$ is small, one unit of y -Brownian time consumes a large calendar time, so $\mathfrak{m}_\eta$ becomes large.³⁹ To see this, start from the fact that by (S.1.1), $\mathcal{L}_\eta \psi(q)$ is the instantaneous mean change of $\psi(q_t)$ when the state equals q . If q_t has a Lebesgue density φ , then

$$\frac{d}{dt} \mathbb{E}[\psi(q_t)] = \int_0^{u_\eta} (\mathcal{L}_\eta \psi) \varphi dq.$$

A stationary density makes the left-hand side zero for every compactly supported ψ . From the scale-speed form (S.1.5), $\mathcal{L}_\eta \psi \cdot \mathfrak{m}_\eta$ is an exact q -derivative, so

$$\int_0^{u_\eta} (\mathcal{L}_\eta \psi) \mathfrak{m}_\eta dq = \int_0^{u_\eta} \partial_q \left(\frac{\psi'}{\mathfrak{s}'_\eta} \right) dq = 0$$

³⁸Note that neither $\mathfrak{m}_\eta$ nor $\mathfrak{m}^Y$ integrates to one yet.

³⁹For compactly supported C^2 test functions ψ and ϕ vanishing near the endpoints, we also have

$$\int_0^{u_\eta} (\mathcal{L}_\eta \psi) \phi \mathfrak{m}_\eta dq = \int_0^{u_\eta} \psi (\mathcal{L}_\eta \phi) \mathfrak{m}_\eta dq.$$

whenever ψ vanishes near the endpoints. Every multiple of $\mathbf{m}_\eta$ therefore keeps these expectations still. When $\int_0^{u_\eta} \mathbf{m}_\eta$ is finite, dividing by that total mass produces a probability density, and the probability of a Borel set A is the fraction of calendar time a long path spends in A .

Substituting the scale density in (S.1.3) into (S.1.4), we obtain:

$$\mathbf{m}_\eta(q) = \frac{2}{\zeta_\eta(q)^2 e^{-\mathcal{R}_\eta(q)}} = \frac{2e^{\mathcal{R}_\eta(q)}}{\zeta_\eta(q)^2}.$$

The integrand of N_η in (3.9) is thus $\mathbf{m}_\eta/2$. Finite total speed mass is the same statement as $N_\eta < \infty$, and the factor two cancels if one normalizes $\mathbf{m}_\eta$ to a probability.⁴⁰

The pairing shows that every multiple of $\mathbf{m}_\eta$ is a stationary candidate. The forward equation then lists every feasible stationary density. It starts with the evolution of the law of q_t . Write $\varphi_\eta(t, q)$ for a smooth density of the current state on $(0, u_\eta)$. Then $\mathbb{E}[\mathcal{L}_\eta \psi(q_t)] = \int (\mathcal{L}_\eta \psi) \varphi_\eta dq$. Differentiating $\mathbb{E}[\psi(q_t)]$ in calendar time also gives $\int \psi \partial_t \varphi_\eta dq$. Integration by parts against compactly supported test functions converts the identity into the Kolmogorov forward, or Fokker-Planck, continuity equation given by

$$\begin{aligned} \partial_t \varphi_\eta(t, q) &= -\partial_q \mathbf{j}[\varphi_\eta](t, q), \\ \mathbf{j}[\varphi_\eta] &= \mu_\eta \varphi_\eta - \frac{1}{2} \partial_q (\zeta_\eta^2 \varphi_\eta). \end{aligned} \tag{S.1.7}$$

The current $\mathbf{j}[\varphi_\eta]$ is the net probability flux through q . On the open interval the identity is the evolution of the law up to exit.

A stationary density is time-independent, so $\partial_q \mathbf{j} = 0$ and the current equals a constant $\mathbf{j}$. The stationary forward equation is therefore the first-order identity

$$\mu_\eta(q) \varphi(q) - \frac{1}{2} \partial_q (\zeta_\eta(q)^2 \varphi(q)) = \mathbf{j}.$$

If the current vanishes with $\mathbf{j} = 0$, this rearranges as $\partial_q (\zeta_\eta^2 \varphi) = 2\mu_\eta \varphi$, or

$$\partial_q \log(\zeta_\eta(q)^2 \varphi(q)) = \frac{2\mu_\eta(q)}{\zeta_\eta(q)^2}.$$

Integrating from q_0 gives $\zeta_\eta^2 \varphi \propto e^{\mathcal{R}_\eta}$. Therefore, the zero-current member of the list of density is a multiple of $e^{\mathcal{R}_\eta}/\zeta_\eta^2$, hence a multiple of $\mathbf{m}_\eta$.

For a non-zero constant current $\mathbf{j}$, the same first-order ODE becomes inhomogeneous. The above $\mathbf{j} = 0$ solution $e^{\mathcal{R}_\eta}/\zeta_\eta^2$ is already known, so write every candidate as that profile times

⁴⁰Infinite scale distance is sufficient for unattainability, and the entrance test is the nested integral of (S.1.3) and (S.1.4). Both belong to the Feller classification in Helland (1996, Sections 2 and 5, Theorem 8).

an unknown function χ of the state,

$$\varphi(q) = e^{\mathcal{R}_\eta(q)} \zeta_\eta(q)^{-2} \chi(q).$$

Here χ is the C^1 multiplier that remains after the homogeneous solution has been factored out. If χ is constant, we recover a multiple of $\mathbf{m}_\eta$. If χ varies, it accounts for $\mathbf{j} \neq 0$. Differentiating the product $\zeta_\eta^2 \varphi = e^{\mathcal{R}_\eta} \chi$ and using $\partial_q \mathcal{R}_\eta = 2\mu_\eta / \zeta_\eta^2$ gives

$$\partial_q(\zeta_\eta^2 \varphi) = e^{\mathcal{R}_\eta} \left(\frac{2\mu_\eta}{\zeta_\eta^2} \chi + \chi' \right).$$

Substituting into the stationarity condition $\mu_\eta \varphi - (\partial_q(\zeta_\eta^2 \varphi))/2 = \mathbf{j}$, the terms that contain χ cancel, and what remains is

$$-\frac{1}{2} e^{\mathcal{R}_\eta} \chi' = \mathbf{j}.$$

Hence $\chi' = -2\mathbf{j} e^{-\mathcal{R}_\eta}$. But $e^{-\mathcal{R}_\eta} = \mathbf{s}'_\eta$, so $\chi' = -2\mathbf{j} \mathbf{s}'_\eta$ and

$$\chi(q) = C_0 - 2\mathbf{j} \mathbf{s}_\eta(q)$$

for an integration constant C_0 . Therefore $\varphi = (e^{\mathcal{R}_\eta} / \zeta_\eta^2)(C_0 - 2\mathbf{j} \mathbf{s}_\eta)$. Since $\mathbf{m}_\eta = 2e^{\mathcal{R}_\eta} / \zeta_\eta^2$, this is

$$\varphi(q) = \mathbf{m}_\eta(q) [C_{\text{stat}} - \mathbf{j} \mathbf{s}_\eta(q)], \quad (\text{S.1.8})$$

with $C_{\text{stat}} = C_0/2$. The speed density $\mathbf{m}_\eta$ is positive on $(0, u_\eta)$, so the sign of φ is the sign of $C_{\text{stat}} - \mathbf{j} \mathbf{s}_\eta$. By construction $\mathbf{s}_\eta(q_0) = 0$ and $\mathbf{s}'_\eta > 0$, so $\mathbf{s}_\eta$ is continuous and strictly increasing and takes every value between its endpoint limits. If those limits are $\pm\infty$, i.e., $\mathbf{s}_\eta(0^+) = -\infty$ and $\mathbf{s}_\eta(u_\eta^-) = \infty$, a non-zero $\mathbf{j}$ makes $C_{\text{stat}} - \mathbf{j} \mathbf{s}_\eta$ change sign and φ cannot stay non-negative. It is verified in Appendix A.4, meaning that only $\mathbf{j} = 0$ solution, i.e., (3.9), is viable.

These properties are the input to the boundary arguments of Appendix A.4. Infinite scale distance from an interior point makes the corresponding endpoint inaccessible. Finite speed mass near an endpoint makes its contribution to N_η finite, and therefore makes consumption utility integrable when log-consumption varies with $\log q$. The nested integral of scale and speed constitutes the entrance test: it diverges at $q = 0$ and converges at $q = u_\eta$. After those integrals are in hand, the same proof returns to (S.1.8) and rules out a non-zero $\mathbf{j}$, because the scale function constructed here tends to $-\infty$ at the origin and to $+\infty$ at u_η .

S.2 Lipschitz extension of the state coefficients

The boundary classification in Appendix A.4 describes the behavior of (3.6) near end-points 0 and u_η . The recurrence argument there also applies Itô's formula to a Lyapunov function that diverges at both endpoints. Both steps require the drift μ_η and the diffusion loading ζ_η to extend Lipschitz-continuously from $(0, u_\eta)$ to the compact interval $[0, u_\eta]$. This section checks whether that extension is allowed. The coefficients are already C^∞ on the open interval.

The only obstruction is $q \downarrow 0$. Differentiating (3.4) gives (A.9):

$$F'_\eta(q) = -\frac{\kappa(m-\nu)}{\beta\nu}q^{m-\nu-1} - \frac{(\varepsilon-1)\beta}{2}q^{\nu-1} - \phi_y - (\phi_\pi - 1)\beta\nu q^{\nu-1}.$$

The exponent $\nu - 1$ is negative as $0 < \nu < 1$. Thus $F'_\eta(q)$ may tend to $-\infty$, and $(\sigma_\eta^s)'$ diverges with it. The state coefficients $\mu_\eta(q)$ and $\zeta_\eta(q)$, however, always carry an extra factor of q . By multiplying (A.9) by q , we get

$$qF'_\eta(q) = -\frac{\kappa(m-\nu)}{\beta\nu}q^{m-\nu} - \frac{(\varepsilon-1)\beta}{2}q^\nu - \phi_y q - (\phi_\pi - 1)\beta\nu q^\nu.$$

The shape restrictions give $m - \nu > 0$ and $\nu > 0$, so every power on the right-hand side tends to 0. Hence $qF'_\eta(q) \rightarrow 0$. The same holds for $H(q)$. From (3.3), $H'(q) = \phi_y + (\phi_\pi - 1)\beta\nu q^{\nu-1}$, and

$$qH'(q) = \phi_y q + (\phi_\pi - 1)\beta\nu q^\nu \rightarrow 0.$$

The volatility derivative can be obtained via the chain rule through F_η . As in the lower-tail expansion of Appendix A.4, $d\sigma_\eta^s/dF_\eta = 1/(\sigma + (1 + \nu)\sigma_\eta^s)$, so

$$(\sigma_\eta^s)'(q) = \frac{F'_\eta(q)}{\sigma + (1 + \nu)\sigma_\eta^s(q)}, \quad q(\sigma_\eta^s)'(q) = \frac{qF'_\eta(q)}{\sigma + (1 + \nu)\sigma_\eta^s(q)}.$$

The denominator is at least $\sigma > 0$, and the numerator tends to 0, so $q(\sigma_\eta^s)'(q) \rightarrow 0$. The function $(\sigma_\eta^s)'$ itself may still be unbounded. Only the product with q remains finite.

Now differentiate the state coefficients. Write $\mu_\eta(q) = q[H(q) + \sigma\sigma_\eta^s(q) + (\sigma_\eta^s(q))^2]$. The product rule gives

$$\mu'_\eta(q) = H(q) + \sigma\sigma_\eta^s(q) + (\sigma_\eta^s(q))^2 + qH'(q) + \sigma q(\sigma_\eta^s)'(q) + 2\sigma_\eta^s(q) q(\sigma_\eta^s)'(q).$$

Send $q \downarrow 0$. The first three terms tend to $-\Phi + \sigma\bar{\sigma}^s + (\bar{\sigma}^s)^2$, which is finite. The remaining three terms each contain a factor qH' or $q(\sigma_\eta^s)'$ and tend to 0, using $\sigma_\eta^s \rightarrow \bar{\sigma}^s$. Therefore

$\mu'_\eta(0+)$ exists and equals $-\Phi + \sigma\bar{\sigma}^s + (\bar{\sigma}^s)^2$. For the diffusion loading $\zeta_\eta(q) = q\sigma_\eta^s(q)$,

$$\zeta'_\eta(q) = \sigma_\eta^s(q) + q(\sigma_\eta^s)'(q).$$

The first summand tends to $\bar{\sigma}^s$ and the second to 0, so $\zeta'_\eta(0+) = \bar{\sigma}^s$. In particular both one-sided derivatives at 0 are finite.

At the upper endpoint u_η , no divergence problem appears. The root u_η lies in $(0, 1)$, where F_η is C^∞ , and $\sigma > 0$ keeps $\sqrt{\sigma^2 + 2(1 + \nu)F_\eta}$ smooth in a neighborhood of u_η inside $(0, 1)$. Thus σ_η^s , μ_η and ζ_η are all C^1 up to u_η . The derivatives μ'_η and ζ'_η are therefore bounded on the compact interval $[0, u_\eta]$. The mean value theorem then makes μ_η and ζ_η Lipschitz there.

The proof of Proposition 3 uses this extension twice: local strong existence and uniqueness run up to exit from $(0, u_\eta)$, and the Lyapunov function of the recurrence argument may be differentiated at states arbitrarily close to either endpoint.

S.3 Lamperti coordinate and Girsanov comparison

Let us return to the recurrence argument in Appendix A.4, at the point where a Lyapunov sublevel C^{41} has been enclosed in a strictly larger compact interval $I \subset (0, u_\eta)$. This section changes coordinates on I . In the new coordinate, Girsanov's theorem (Karatzas and Shreve, 1991) supplies the Radon-Nikodym derivative of the physical law relative to Brownian motion, and that weight becomes bounded below on paths that remain in I .

On I , the diffusion loading ζ_η is continuous and bounded away from zero, but it depends on q . The Lamperti coordinate (Iacus, 2008, Section 1.11.4) removes that state dependence. Let $q_o \in I$ be the interior reference point already used in (3.9), and set

$$x(q) = \int_{q_o}^q \zeta_\eta(\tilde{q})^{-1} d\tilde{q}.$$

Then $x'(q) = 1/\zeta_\eta(q)$ and $x''(q) = -\zeta'_\eta(q)/\zeta_\eta(q)^2$. On I , the map $q \mapsto x(q)$ is a C^2 diffeomorphism onto a compact x -interval I_x . Write $x(C) = \{x(q) : q \in C\}$ for the image of the sublevel. The map is continuous and strictly increasing, so $x(C)$ is a compact subinterval of the interior of I_x . It is the same set of states as C , read on the unit-diffusion axis rather than on the q -axis. Write $x_t = x(q_t)$. Itô's formula gives

$$dx_t = x'(q_t) dq_t + \frac{1}{2} x''(q_t) \zeta_\eta(q_t)^2 dt = \left(\frac{\mu_\eta(q_t)}{\zeta_\eta(q_t)} - \frac{1}{2} \zeta'_\eta(q_t) \right) dt + dZ_t.$$

⁴¹In Appendix A.4, a candidate C is a sublevel of the Lyapunov function (A.35), $\{\mathcal{V}_{\delta,k} \leq R\}$. Properness of $\mathcal{V}_{\delta,k}$ makes the sublevel a compact subset of $(0, u_\eta)$.

The diffusion loading becomes identically one. The remaining drift is a function of the current x -state. Define

$$\mathbf{b}(x) = \frac{\mu_\eta(q)}{\zeta_\eta(q)} - \frac{1}{2}\zeta'_\eta(q),$$

evaluated at the unique $q \in I$ with $x(q) = x$. Continuity of μ_η and C^1 regularity of ζ_η on I make $\mathbf{b}$ continuously differentiable on I_x . Thus, while the path remains in I ,

$$dx_t = \mathbf{b}(x_t) dt + dZ_t, \tag{S.3.1}$$

which is Brownian motion plus a drift. Note that with $\mathbf{b}(x_t)$ present, the transition density of (S.3.1) on I_x is not a Gaussian, so its strict positivity is still unknown. (S.3.1) with $\mathbf{b}(x_t) \equiv 0$ is Brownian motion on I_x , implying its transition density is a Gaussian and is strictly positive between interior endpoints. That is where Girsanov's theorem (Karatzas and Shreve, 1991) is useful as it removes $\mathbf{b}(x_t)$ by reweighting that Brownian path measure. Under the reweighted measure the paths follow (S.3.1).

Reweightings Let W be a Wiener process on the x -axis. Define P as the law (i.e., probability measure) of $x_s = x_0 + W_s$ stopped at the exit from I_x .⁴² Define Q_{phys} as the *physical* law of (S.3.1) stopped at the same exit. Girsanov's theorem allows us reweight P so that the new paths solve (S.3.1), and we use uniqueness to identify the reweighted measure with Q_{phys} .

Under P -measure, the process is $x_s = x_0 + W_s$ until exit.⁴³ On compact I_x , $\mathbf{b}$ is bounded, so Novikov's condition (Novikov, 1972) holds under P and

$$\mathcal{E}_t = \exp \left(\int_0^t \mathbf{b}(x_s) dW_s - \frac{1}{2} \int_0^t \mathbf{b}(x_s)^2 ds \right)$$

is a true martingale under P , with $E_P[\mathcal{E}_\Delta] = 1$. The quadratic correction $-\frac{1}{2} \int \mathbf{b}^2$ is the Itô term that removes drift from $\mathcal{E}_t$ itself. Indeed, if $Y_t = \int_0^t \mathbf{b} dW - \frac{1}{2} \int_0^t \mathbf{b}^2 ds$, then $dY = \mathbf{b} dW - \frac{1}{2} \mathbf{b}^2 dt$ and $(dY)^2 = \mathbf{b}^2 dt$, so

$$d\mathcal{E}_t = e^{Y_t} dY_t + \frac{1}{2} e^{Y_t} (dY_t)^2 = \mathcal{E}_t \mathbf{b}(x_t) dW_t.$$

Thus, $\mathcal{E}$ is a local martingale under P .⁴⁴ Boundedness of $\mathbf{b}$ upgrades the P -local martingale to a P -martingale. If we construct another measure Q defined by $dQ = \mathcal{E}_\Delta dP$, then Q becomes

⁴²This stopped path is called the killed process in Appendix A.4. The position is no longer recorded from the first exit of I , or of its image I_x . The same exit stops Q_{phys} .

⁴³Note that the physical shock Z is not used on this space.

⁴⁴Note that it is not a local martingale under Q_{phys} .

therefore a probability measure on path space.⁴⁵

Note that the definition $dQ = \mathcal{E}_\Delta dP$ does not by itself declare any process to be Brownian motion under Q . It only changes how paths are counted. Lévy's characterization ([Karatzas and Shreve, 1991](#), Section 3.3, Theorem 3.16) is then used to check a candidate for Brownian motion under dQ . Write

$$\widehat{W}_t = W_t - \int_0^t \mathbf{b}(x_s) ds.$$

Recall that quadratic variation is calculated path-wise, and does not depend on measures. Subtracting a finite-variation integral does not change it, so $[\widehat{W}, \widehat{W}]_t = [W, W]_t = t$ under both P and Q . Continuity is likewise path-wise. Therefore $\widehat{W}$ is a Q -Brownian motion as soon as it is a Q -local martingale.

A process N is a Q -local martingale if and only if the product $N\mathcal{E}$ is a P -local martingale by construction. Note that W is Brownian motion and $d\mathcal{E}_t = \mathcal{E}_t \mathbf{b}(x_t) dW_t$ has already been obtained under P . The product rule for $\widehat{W}_t \mathcal{E}_t$ gives

$$d(\widehat{W}_t \mathcal{E}_t) = \mathcal{E}_t (\widehat{W}_t \mathbf{b}(x_t) + 1) dW_t,$$

implying that $\widehat{W}\mathcal{E}$ is a P -local martingale, which in turn implies $\widehat{W}$ is a Q -local martingale. Combined with $[\widehat{W}, \widehat{W}]_t = t$, Lévy gives that $\widehat{W}$ is a Q -Brownian motion.

Under P , we have $dx_t = dW_t$. Substituting $dW_t = d\widehat{W}_t + \mathbf{b}(x_t) dt$ thus gives, under Q ,

$$dx_t = \mathbf{b}(x_t) dt + d\widehat{W}_t$$

until exit from I_x . Thus, every Q -typical path solves the same killed SDE as the physical process ([S.3.1](#)). On I_x , the drift $\mathbf{b}$ is C^1 , hence Lipschitz, so ([S.3.1](#)) has a unique weak solution ([Karatzas and Shreve, 1991](#), Proposition 3.20).⁴⁶ Thus, $Q = Q_{\text{phys}}$, implying

$$Q_{\text{phys}}(A) = E_P[1_A \mathcal{E}_\Delta].$$

The Radon–Nikodym derivative dQ_{phys}/dP is the constructed factor $\mathcal{E}_\Delta$. Equivalently, the killed physical law (Q_{phys}) becomes the killed Brownian law (P) multiplied by $\mathcal{E}_\Delta$. The killed, drifted transition density of ([S.3.1](#)) then equals the killed Brownian (Gaussian) density times the conditional expectation of $\mathcal{E}_\Delta$ given the endpoints. A strictly positive lower bound on $\mathcal{E}_\Delta$ along surviving paths with endpoints in $x(C)$ will therefore transfer a lower bound from the Brownian density to the drifted density.

⁴⁵At this stage Q is only a constructed measure. It will eventually be identified with Q_{phys} .

⁴⁶Under unique weak existence, any two measures on paths under which the process solves the same SDE must be the same measure.

Let $\mathfrak{B}$ be an anti-derivative of $\mathfrak{b}$, so that $\mathfrak{B}'(x) = \mathfrak{b}(x)$ and $\mathfrak{B}''(x) = \mathfrak{b}'(x)$. On a surviving reference path and under P-measure, Itô's formula applied to $\mathfrak{B}(x_s)$ uses $dx_s = dW_s$ and $(dx_s)^2 = ds$:

$$d\mathfrak{B}(x_s) = \mathfrak{B}'(x_s) dx_s + \frac{1}{2} \mathfrak{B}''(x_s) ds = \mathfrak{b}(x_s) dW_s + \frac{1}{2} \mathfrak{b}'(x_s) ds.$$

Integrate from 0 to Δ :

$$\mathfrak{B}(x_\Delta) - \mathfrak{B}(x_0) = \int_0^\Delta \mathfrak{b}(x_s) dW_s + \frac{1}{2} \int_0^\Delta \mathfrak{b}'(x_s) ds.$$

Solve for the stochastic integral:

$$\int_0^\Delta \mathfrak{b}(x_s) dW_s = \mathfrak{B}(x_\Delta) - \mathfrak{B}(x_0) - \frac{1}{2} \int_0^\Delta \mathfrak{b}'(x_s) ds.$$

Substitute this identity into the exponent of $\mathcal{E}_\Delta$:

$$\int_0^\Delta \mathfrak{b} dW - \frac{1}{2} \int_0^\Delta \mathfrak{b}^2 ds = \mathfrak{B}(x_\Delta) - \mathfrak{B}(x_0) - \frac{1}{2} \int_0^\Delta [\mathfrak{b}'(x_s) + \mathfrak{b}(x_s)^2] ds.$$

Therefore

$$\mathcal{E}_\Delta = \exp \left(\mathfrak{B}(x_\Delta) - \mathfrak{B}(x_0) - \frac{1}{2} \int_0^\Delta [\mathfrak{b}'(x_s) + \mathfrak{b}(x_s)^2] ds \right). \quad (\text{S.3.2})$$

On paths that start and end in $x(C)$ and never leave I_x , the likelihood function $\mathcal{E}_\Delta$ in (S.3.2) is bounded below by a positive constant $\varepsilon_\mathcal{E}$. Indeed, $\mathfrak{B}$ is continuous on the compact I_x , so

$$\mathfrak{B}(x_\Delta) - \mathfrak{B}(x_0) \geq -2 \sup_{I_x} |\mathfrak{B}|.$$

$\mathfrak{b}' + \mathfrak{b}^2$ is also continuous on I_x , hence bounded by some finite $M_\mathfrak{b}$. The time integral over an interval of length at most Δ is therefore at most $M_\mathfrak{b}\Delta$, and⁴⁷

$$\mathcal{E}_\Delta \geq \exp \left(-2 \sup_{I_x} |\mathfrak{B}| - M_\mathfrak{b}\Delta/2 \right) =: \varepsilon_\mathcal{E} > 0.$$

Appendix A.4's equation (A.43) uses this bound. It multiplies $\varepsilon_\mathcal{E}$ by a strictly positive lower bound on the killed Brownian density between endpoints in $x(C)$, changes variables back to the output ratio q , and obtains the minorization constant ε_C on the original sublevel C .

⁴⁷The right-hand side depends on Δ , C and I only through the compact I_x .

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