Higher-Order Forward Guidance[*]

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Abstract

This paper provides a model of the business cycle that incorporates financial markets and endogenous financial volatility at the Zero Lower Bound (ZLB). Within this framework, forward guidance is identified as a crucial mechanism for coordinating the actions of market participants, guiding the economy towards optimal equilibrium paths with lower financial volatility and enhanced welfare. We reveal two novel insights: (i) Central banks, by credibly pledging future economic stabilization, can mitigate excess financial market volatility at the ZLB; (ii) Alternatively, a central bank's commitment not to stabilize the economy in the future can direct the economy towards more favorable equilibrium paths with reduced endogenous volatility at the ZLB, thus presenting an interesting trade-off between future business cycle stabilization and reduced financial volatility at the ZLB. Finally, an examination of alternative fiscal policies reveals that measures aimed at encouraging increased investment in risky assets can stimulate economic activity at the ZLB by positively impacting aggregate household financial wealth.

Keywords: Monetary Policy, Forward Guidance, Macroprudential Policy, Financial Volatility, Risk-Premium

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1 Introduction

In the aftermath of the Great Recession and the recent Covid-19 pandemic, prolonged periods of constrained policy rates at the Zero Lower Bound (ZLB) have underscored the need for alternative monetary interventions, notably forward guidance. ZLB episodes are often characterized by heightened financial market volatility, exacerbated by the reduced efficacy of conventional monetary policy tools. In this context, forward guidance goes beyond its traditional roles of conveying economic forecasts (delphic guidance) and making policy commitments (odyssean guidance), and evolves into a tool for coordinating market participant actions and reducing overall economic uncertainty. This paper provides an analytically tractable framework for examining the effects of unconventional policies at the ZLB, and especially investigates the impact of forward guidance on the volatility of financial markets and the business cycle as well as welfare.

Our paper builds on the model which integrates endogenous financial volatility within a Two-Agent New Keynesian (TANK) framework. The model features a representative stock market index that encapsulates the ownership rights to the profits of firms in the economy. A group of hand-to-mouth workers supplies labor to these firms, while a group of capitalists holds the economy's aggregate financial wealth, allocating it between consumption and portfolio choices. In equilibrium, the wealth of capitalists is directly affected by the stock market performance. In this environment, an increase in endogenous financial volatility raises market risk-premium, leading to depressed asset prices and wealth of capitalists and lowering in turn aggregate demand, whose fluctuation determines the endogenous financial volatility itself.[1] This dynamic creates a coordination challenge for economic agents and might lead to self-fulfilling shocks in volatility, resulting in an endogenous state of elevated financial volatility. While Lee and Dordal i Carreras (2023) investigates the determinacy of the model's solutions under conventional monetary policy regimes in an nonlinear New Keynesian environment, this paper focuses on whether central bank forward guidance can steer agents towards equilibrium paths with lower financial volatility and quicker economic stabilization times at the ZLB.

Our analysis begins by exploring whether financial volatility intensifies when conventional monetary policy is constrained by the ZLB. We discover that a credible commitment from the central bank to stabilize the economy *after* the ZLB period can also ensure that

[1] Again, a decline in aggregate demand leads to reduced firm profitability, negatively impacting both the stock market capitalization and the aggregate wealth of capitalists. So in our framework, economic volatility and financial market volatility are tightly connected.

the excess volatility does not appear *during* the ZLB. This conclusion is derived through backward induction: if the monetary authority credibly commits to stabilize the economy in a finite period of time, it rules out the possibility of catastrophic (or exuberant) scenarios that contribute to the economic volatility faced by the agents. As a result, this precludes the feasibility of the unfavorable coordination equilibrium paths that would initially lead to these scenarios.

We then analyze the benefits of various forward guidance strategies. In our framework, traditional forward guidance includes an Odyssean component, in which the central bank credibly commits to keeping the policy rate at zero for a period longer than what economic conditions minimally require. Following this extended ZLB period of Odyssean guidance, the central bank implements a policy rule aimed at perfect stabilization outside the ZLB, as assumed in the preceding paragraph. The outcomes align with those identified in the prior research: by committing to a future period of accommodative policy rates, the central bank implicitly agrees to a temporary phase of positive excess demand and profits. This effect, owing to the forward-looking nature of stock markets, positively influences stock values at present, thereby raising aggregate demand during the ZLB. Such an approach spreads the costs of the ZLB over time, and is preferred when considering the quadratic welfare costs of fluctuations in the output gap. In addition, the commitment to perfect stabilization in the future precludes the appearance of excess financial volatility at the ZLB, as previously discussed.

The next strategy we consider explicitly leverages the agents' coordination problem to direct them towards an equilibrium with reduced financial and economic volatility at the ZLB. We term this approach *higher-order forward guidance*. For its execution, the central bank must relinquish the promise of perfect stabilization in the future: by committing not to enforce perfect stabilization at the conclusion of the Odyssean guidance period, the central bank makes possible the existence of coordinated equilibriums that were previously ruled out by backward induction. This strategy allows the central bank to guide agents towards equilibrium paths with low levels of volatility and risk-premium at the ZLB, thereby maximizing expected welfare beyond the capabilities of traditional forward guidance (which we identify as a limiting case of this strategy). However, this intervention has its trade-offs: by committing not to stabilize the business cycle after the ZLB period, the central bank risks significant future output gap deviations. Thus, our higher-order guidance weighs the lack of stabilization in the future economy against reduced financial volatility in the present while at the ZLB. Furthermore, we uncover that even the central bank's slight hint that perfect stabilization is not guaranteed at the conclusion of the Odyssean guidance period allows our higher-order forward guidance strategy to be viable.[2]

Finally, we analyze two macroprudential policies at the ZLB designed to incentivize investors to assume more financial risk, thereby raising asset prices and aggregate demand: (i) a subsidy on risky asset investments (or equivalently, a reduction in capital gains taxes), and (ii) fiscal redistribution among agents. The first policy illustrates that a temporary subsidy on holding risky assets at the ZLB enhances their Sharpe ratio, leading to higher asset prices and increased aggregate financial wealth of the economy. This surge in financial wealth boosts the aggregate demand of capitalists and alleviates the severity of recessions, as well as the welfare costs associated with the ZLB. However, our study emphasizes the need to consider the varying marginal propensities to consume (MPC) across households when selecting the optimal funding sources for the subsidy. In a hypothetical scenario where the subsidy is financed by non-distortionary taxation on hand-to-mouth workers, the policy's effectiveness is completely nullified: the increase in financial wealth and aggregate demand induced by the subsidy is exactly offset by a reduction in workers' consumption, which negatively impacts firm profitability and stock market capitalization. In this context, the second macroprudential policy examined focuses on the effects of fiscal transfers at the ZLB from capitalists with a low marginal propensity to consume to hand-to-mouth workers with a high marginal propensity to consume. As expected, this transfer leads to an increase in the economy's aggregate demand. We contribute to the literature by showing that such redistribution fosters increased demand through another channel: the initial rise in demand from workers' consumption boosts firm profitability, which in turn increases the financial wealth of capitalists and their willingness to invest in risky assets, as well as consumption demand, again raising financial wealth and initiating a self-reinforcing cycle.

Featuring a demand-driven economy with perfectly rigid prices,[3] our framework emphasizes the significant impact of stock market performance on aggregate demand. Unlike prior studies, such as Akerlof and Yellen (1985), Blanchard and Kiyotaki (1987), Eggerts-

[2]To be specific, we prove that if the central bank promises there is a tiny probability that the business cycle might not be stabilized at the end of Odyssean forward guidance, the higher-order forward guidance strategy becomes viable. Our model features a novel discontinuity in that regard: if the monetary authority achieves perfect stabilization with certainty after the ZLB period, we return to the traditional forward guidance case in which no excess volatility or risk premium is manipulated by the central bank. Even with a slight chance that the perfect stabilization is relinquished, the central bank can engineer a better equilibrium with lower levels of financial volatility and risk-premium based on our higher-order forward guidance strategy.

[3]This assumption simplifies the analysis. An extended model with sticky prices à la Calvo (1983) produces qualitatively similar results.

son and Krugman (2012), Farhi and Werning (2012, 2016, 2017), Korinek and Simsek (2016), and Schmitt-Grohé and Uribe (2016), who focus on demand-driven recessions due to deleveraging borrowers and aggregate demand externalities, our model triggers the ZLB episodes with a decrease in aggregate demand for risky assets, identified as a key driver of financial recessions by Caballero and Farhi (2017) and Caballero and Simsek (2020). In a similar way to Werning (2012), we assume the economy's exogenous and deterministic shift to the ZLB, here resulting from a shock that raises the risk premium in financial markets and reduces the demand for risky assets, resulting in a downward jump in the natural rate of interest to a negative territory. Our approach diverges from the literature by including an endogenous component to financial volatility, influenced by both the ZLB and forward guidance. Papers including Eggertsson et al. (2003), Campbell et al. (2012, 2019), Del Negro et al. (2013), McKay et al. (2016), and Caballero and Farhi (2017) explore the implications of forward guidance at the ZLB from both theoretical and empirical perspectives. Our research distinguishes itself by focusing specifically on the impact of forward guidance on higher-order moments including the endogenous volatility of financial markets and the broader economy.[4] In addition, our study of macroprudential policies at the ZLB, while building on the existing literature, e.g., Lorenzoni (2008), Farhi and Werning (2012, 2016, 2017), and Korinek and Simsek (2016), places a stronger emphasis on the interplay between asset prices and aggregate demand.

The structure of this paper is organized as follows: Section 2 presents the model. Section 3 discusses the incorporation of the ZLB into our framework. Section 4 examines the effectiveness of various forward guidance strategies. Section 5 studies other macroprudential policies at the ZLB. Section 6 provides concluding remarks.[5]

2 The Model

We begin by introducing a theoretical framework that facilitates the analysis of higher-order moments related to the aggregate financial and economic volatility of the economy.[6]

[4]Our approach, where central bank communications serve as an equilibrium coordination device, aligns well with the concept of 'open-mouth' operations at the ZLB described by Campbell and Weber (2019).

[5]Appendix I contains the parameter calibration, and derivations and proofs are detailed in Appendix II. Online Appendix contains additional derivations and proofs.

[6]Our results except those in Section 5 hold in a non-linear version of the standard New Keynesian model as the model would feature the first-order impacts of the aggregate endogenous volatility as we do here. We choose this model as it elucidates the interaction among financial volatility, aggregate wealth, and aggregate demand, and allows us to study various macroprudential policies in a tractable way, as we do in Section 5.

2.1 Setting

We consider a continuous-time framework within a filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\in\mathbb{R}}, \mathbb{P})$. The economy is composed of two equally sized agent groups: capitalists, characterized as neoclassical agents, and hand-to-mouth workers, conceptualized as Keynesian agents. This structure, closely aligned with the approach of Greenwald et al. (2014), assumes that all financial wealth is held by capitalists, while workers rely on labor income for consumption. The aggregate technology, denoted by A_t, introduces a single source of exogenous variation in the model and generates the filtration $(\mathcal{F}_t)_{t\in\mathbb{R}}$. The process evolves according to a geometric Brownian motion given by:

$$\frac{dA_t}{A_t} = \underbrace{g}_{\text{Growth}} dt + \underbrace{\sigma_t}_{\text{Fundamental risk}} dZ_t \ ,$$

where g represents the expected growth rate, and σ_t signifies the economy's *fundamental* risk, which we take as exogenous. For simplicity, σ_t is initially assumed constant and equal to σ in Section 2. Later, in Section 3, we introduce a deterministic shift in σ_t to explore various scenarios involving the ZLB.

2.1.1 Firms

The economy features a unit measure of monopolistically competitive firms, each producing a unique intermediate good $y_t(i)$, for $i \in [0, 1]$. These intermediate firms contribute to the final good y_t through a Dixit-Stiglitz aggregation function with a substitution elasticity $\epsilon > 0$, as given by:

$$y_t = \left(\int_0^1 y_t(i)^{\frac{\epsilon-1}{\epsilon}} di \right)^{\frac{\epsilon}{\epsilon-1}} .$$

Each intermediate firm i employs a production function $y_t(i) = A_t(N_{W,t})^{\alpha} n_t(i)^{1-\alpha}$, where $N_{W,t}$ is the total labor in the economy, and $n_t(i)$ is the labor demand of firm i at time t. The inclusion of a production externality à la Baxter and King (1991) helps to align our model with observed asset price and wage co-movements, and does not alter other qualitative outcomes of our model.[7]

[7] In a model without Baxter and King (1991) externalities, increasing asset prices often correlate with lower wages, which is contrary to the empirical evidence (Chodorow-Reich et al., 2021) regarding the effects of stock price hikes on aggregate demand, employment, and wages. The Baxter and King (1991) externality

Intermediate firms face a downward-sloping demand curve $y_i(p_t(i)|p_t, y_t)$, with $p_t(i)$ representing the price of their own good, and p_t and y_t the aggregate price index and output, respectively:

$$y_i(p_t(i)|p_t, y_t) = y_t \left(\frac{p_t(i)}{p_t}\right)^{-\epsilon} \quad ,$$

where price index $p_t = \left(\int_0^1 p_t(i)^{1-\epsilon} di\right)^{\frac{1}{1-\epsilon}}$ aggregates prices $\{p_t(i)\}$ from all intermediate goods. For tractability, we assume perfect price rigidity, $p_t(i) = p_t = \bar{p}$ for all t, i.[8] Thus, each firm produces an equal level of output $y_t(i) = y_t$ for all i, determined by demand.

2.1.2 Workers

A representative hand-to-mouth worker supplies labor to the intermediate firm producers, earning wage income $w_t N_{W,t}$ and spending it entirely on final good consumption. The representative worker maximizes:

$$\max_{C_{W,t}, N_{W,t}} \frac{\left(\frac{C_{W,t}}{A_t}\right)^{1-\varphi}}{1-\varphi} - \frac{(N_{W,t})^{1+\chi_0}}{1+\chi_0} \quad , \quad \text{s.t.} \;\; \bar{p} C_{W,t} = w_t N_{W,t} \;, \tag{1}$$

where $C_{W,t}$, $N_{W,t}$, and w_t stand for consumption, labor supply, and wage, respectively, with χ_0 being the inverse Frisch elasticity of labor supply.[9] Under our rigid price assumption, equilibrium labor demand by each firm i, $\{n_t(i)\}$, aggregates linearly into total labor $N_{W,t}$, resulting in $n_t(i) = N_{W,t}$ for all i. Plugging this finding back into the production function, equilibrium output y_t simplifies to a linear function of total labor, $y_t = A_t N_{W,t}$.

2.1.3 Financial Market and Capitalists

Unlike conventional New-Keynesian models where a representative household owns the economy's firms and receives lump-sum rebated profits, we assume firm profits are capitalized in the stock market through a representative index fund. Capitalists are faced with

enables our calibration to reflect these empirical trends by linking higher asset prices and aggregate demand with increased labor demand and wages.

[8]The alternative assumption of sticky price-resetting à la Calvo (1983) does not significantly alter model dynamics.

[9]Consumption $C_{W,t}$ is normalized by the aggregate TFP A_t due to trending economic growth, a standardization that does not affect our model's qualitative results.

an optimal portfolio allocation problem, deciding between investing in a risk-free bond and the stock index at each moment t.

The aggregate nominal value of the stock index fund is represented by $\bar{p}A_tQ_t$, where Q_t is the normalized real index price. This price is endogenously determined and adapts to filtration $(\mathcal{F}_t)_{t\in\mathbb{R}}$, following the equation:

$$\frac{dQ_t}{Q_t} = \mu_t^q dt + \underbrace{\sigma_t^q}_{\substack{\text{Financial}\\ \text{volatility}}} dZ_t \ ,$$

with μ_t^q and σ_t^q representing the endogenous drift and volatility of the process, respectively. We interpret σ_t^q as a measure of financial uncertainty or disruption. Therefore, aggregate financial wealth $\bar{p}A_tQ_t$ evolves according to a geometric Brownian motion, characterized by a combined volatility of $\sigma + \sigma_t^q$. Notably, σ_t^q, determined in equilibrium, can be either positive or negative, indicating that aggregate real stock market value A_tQ_t might be more (or less) volatile than the technology process, $\{A_t\}$. When σ_t^q is negative, the total volatility $\sigma + \sigma_t^q$ becomes smaller than the fundamental volatility σ.[10]

Alongside the stock market, we introduce a risk-free bond with a nominal interest rate i_t, set by the central bank. Bonds are assumed to be in zero net supply in equilibrium. A unit measure of identical capitalists decides how to allocate their wealth between risk-free bonds and the risky stock index. By holding the later, capitalists earn the profits from the intermediate goods sector, which are distributed as stock dividends, and benefit from stock price revaluations due to changes in A_t and Q_t. Given the competitive nature of financial markets, each capitalist takes the nominal risk-free rate i_t, the expected stochastic stock market return i_t^m, and the total risk level $\sigma + \sigma_t^q$ as given when making portfolio decisions.[11] If a capitalist invests a fraction θ_t of their nominal wealth a_t in the stock market, the total risk borne becomes $\theta_t a_t(\sigma + \sigma_t^q)$ over the interval $[t, t + dt]$. Thus, the portfolio's riskiness is directly proportional to the investment share θ_t in the stock index. Capitalists, being risk-averse, demand a risk-premium compensation $i_t^m - i_t$ for investing in the risky index, which

[10]With $\sigma_t^q < 0$, we observe that $\text{Cov}_t\left(dA_t, dQ_t\right) = \sigma\sigma_t^q A_tQ_tdt < 0$.

[11]The competitive market assumption is crucial in our model for explaining inefficiencies stemming from the aggregate demand externality that each capitalist's financial investment decision imposes on the economy. For more details, see Farhi and Werning (2016).

is determined in equilibrium. A representative capitalist solves the following problem:

$$\begin{aligned} &\max_{C_t,\theta_t} \mathbb{E}_0 \int_0^\infty e^{-\rho t} \log C_t dt \ , \\ &\text{s.t. } \ da_t = \left(a_t \left(i_t + \theta_t(i_t^m - i_t)\right) - \bar{p}C_t\right) dt + \theta_t a_t \left(\sigma + \sigma_t^q\right) dZ_t \ , \end{aligned} \tag{2}$$

where ρ and C_t denote the subjective discount rate and final good consumption of capitalists, respectively. At each instant, the capitalist earns returns from both risk-free bond and risky stock investments, allocating their income towards consumption of the final good.

2.2 Equilibrium and Asset Pricing

The nominal state price density of capitalists, denoted as ξ_t^N, can be expressed as follows:

$$\xi_t^N = e^{-\rho t} \frac{1}{C_t} \frac{1}{\bar{p}} \ , \ \text{ where } \ \mathbb{E}_t \left(\frac{d\xi_t^N}{\xi_t^N} \right) = -i_t dt \ , \tag{3}$$

and the stochastic discount factor of capitalists between the present time t and a future time s is defined as $\frac{\xi_s^N}{\xi_t^N}$. The aggregate stock market wealth, $\bar{p}A_tQ_t$, is defined as the sum of discounted profit streams from the intermediate goods sector, priced using ξ_t^N, under the assumption that capitalists are the marginal investors in the stock market in equilibrium.

At time t, the total profit of the intermediate goods sector, denoted as D_t, is given by

$$D_t \equiv \bar{p}y_t - \underbrace{w_t N_{W,t}}_{=\bar{p}C_{W,t}} = \bar{p}(y_t - C_{W,t}) = \bar{p}C_t \ , \tag{4}$$

where $w_t N_{W,t}$, the wage income, is equivalent to the consumption expenditure of hand-to-mouth workers, given by $\bar{p}C_{W,t}$. Consequently, the total dividend is equal to the capitalists' aggregate consumption expenditure. Incorporating equation (4) into the asset pricing equation, we obtain

$$\bar{p}A_tQ_t = \mathbb{E}_t \frac{1}{\xi_t^N} \int_t^\infty \xi_s^N \underbrace{D_s}_{=\bar{p}C_s} ds = \frac{\bar{p}C_t}{\rho} \ , \tag{5}$$

which implies $C_t = \rho A_t Q_t$. It indicates that, in equilibrium, the rate of consumption by capitalists corresponds to a fixed proportion ρ of their aggregate financial wealth. From equations (4) and (5), the dividend yield of the stock market index fund is also constant and equal to ρ, which results in the equilibrium consumption of stock dividends by capitalists.

Agents of the same type (workers or capitalists) are identical and make symmetric decisions in equilibrium. Since bonds have a zero net supply, the capitalists' wealth share in the stock market, denoted as θ_t, must be equal to one for all t. This condition determines the equilibrium risk-premium demanded by capitalists. Drawing on equations (2), (3), and (5), the risk-premium is given by

$$\mathrm{rp}_t \equiv i_t^m - i_t = \left(\sigma + \sigma_t^q\right)^2 \ , \tag{6}$$

where rp_t increases with the total volatility $\sigma + \sigma_t^q$ of the aggregate financial wealth $\bar{p}A_tQ_t$. It is important to note that the wealth gain (or loss) of a capitalist equates to the nominal revaluation of the stock market index. Our equilibrium conditions in equations (5) and (6) are consistent with Merton (1971).

The equilibrium in the goods market and the expected stock return i_t^m are characterized as follows: Given that capitalists' consumption $C_t = \rho A_t Q_t$ holds in equilibrium, the final goods market equilibrium condition can be written as

$$\rho A_t Q_t + \frac{w_t}{\bar{p}} N_{W,t} = y_t = A_t N_{W,t} \ . \tag{7}$$

The nominal expected return on stocks, i_t^m, comprises the dividend yield from firm profits and the nominal stock price revaluation resulting from fluctuations in $\{A_t, Q_t\}$. In equilibrium, changes in i_t^m only affect nominal stock prices, as the dividend yield remains constant and equal to ρ. Defining $\{\mathbf{I}_t^m\}$ as the cumulative stock market return process, where $\mathbb{E}_t\left(dI_t^m\right) = i_t^m dt$, equation (8) decomposes i_t^m into its dividend yield and expected stock revaluation components as follows:

$$\begin{aligned} d\mathbf{I}_t^m &= \frac{\overbrace{\cancel{\bar{p}}\left(\underbrace{y_t - \frac{w_t}{\bar{p}}N_{W,t}}_{=C_t}\right)}^{\text{Nominal dividend}}}{\underbrace{\cancel{\bar{p}}A_tQ_t}_{\text{Total stock market wealth}}}dt + \underbrace{\frac{d\left(\cancel{\bar{p}}A_tQ_t\right)}{\cancel{\bar{p}}A_tQ_t}}_{\text{Stock revaluation}} \\ &= \underbrace{\left(\rho + g + \mu_t^q + \sigma\sigma_t^q\right)}_{=i_t^m}dt + \underbrace{\left(\sigma + \sigma_t^q\right)}_{\text{Risk term}}dZ_t \ . \end{aligned} \tag{8}$$

The real stock price Q_t is a pivotal factor in driving the business cycle in the model's equilibrium. An increase in Q_t leads to the higher consumption of capitalists, leading to higher wages, greater labor demand by firms, and consequently, increased consumption by all households.

Flexible Price Equilibrium In line with most of the literature, we adopt the equilibrium of the flexible price economy as the benchmark that guides the policy goals of the monetary authority. Details of this equilibrium are presented in Online Appendix A. Additionally, Online Appendix B outlines the necessary conditions for positive co-movement among the gaps in asset price, wage, labor supply, and consumption for both capitalists and workers. Here, 'gaps' refer to the log-deviations from the flexible price equilibrium. As illustrated in Online Appendix B, all the gaps are proportional to each other, and hereafter we write equilibrium conditions in asset price gap $\hat{Q}_t$.

In the flexible price equilibrium, denoted by the superscript n (indicating 'natural'), we obtain $\mu_t^{q,n} = \sigma_t^{q,n} = 0$, implying a constant natural stock price, Q_t^n. The natural interest rate, denoted by r_t^n, represents the real risk-free rate in the flexible price economy. In equilibrium, this rate remains constant, and is given by $r^n = \rho + g - \sigma^2$.

2.3 Gap Economy

In particular, we define the risk-premium gap as $\hat{rp}_t \equiv \mathrm{rp}_t - \mathrm{rp}_t^n$, where rp_t^n stands for the natural counterpart of the risk-premium. We also introduce the concept of the risk-adjusted natural rate, r_t^T, defined as:

$$r_t^T \equiv r_t^n - \frac{1}{2}\hat{rp}_t \; . \tag{9}$$

This rate adjusts the natural rate of return to account for the risk differential between rigid and flexible price economies, serving as an anchor for monetary policy in our model. For example, a positive risk-premium gap, $\hat{rp}_t > 0$, reduces the stock market portfolio demand of capitalists compared to the benchmark economy, potentially leading to a recession.

This effect is formally illustrated in equation (10) of Proposition 1, where a decline in r_t^T relative to the risk-free policy rate i_t fosters expectations of future asset price revaluations, which manifest through decreases in current asset prices and output gaps. Note that in a conventional New Keynesian model, the natural rate r_t^n appears in the position of r_t^T in (10).

Proposition 1 (Dynamic IS Equation) *The dynamic IS equation of the model, expressed* *in terms of the asset price gap, is given by:*[12]

$$d\hat{Q}_t = (i_t - r_t^T)dt + \sigma_t^q dZ_t \ , \tag{10}$$

where $r_t^T \equiv r_t^n - \frac{1}{2}\hat{rp}_t$ *is defined as the risk-adjusted natural rate of the economy.*

Proof. See Online Appendix C.

■

2.4 Monetary Policy and Equilibrium Uniqueness

We complete the model by incorporating a monetary policy rule. This rule, in conjunction with the dynamic IS equation defined in equation (10) and the implementation of forward guidance or other macroprudential measures, is necessary to determine the model's solution. The baseline policy rule is expressed as follows:

$$\begin{aligned} i_t &= \max\left\{r_t^T + \phi_q \hat{Q}_t, 0\right\} \\ &= \max\left\{r_t^n - \frac{1}{2}\hat{rp}_t + \phi_q \hat{Q}_t, 0\right\} \ , \end{aligned} \tag{11}$$

where $\phi_q > 0$ satisfies the Taylor principle when not constrained by the ZLB.[13] Combining equations (10) and (11) when the ZLB is not binding, we obtain

$$\mathbb{E}_t \, d\hat{Q}_t = \phi_q \hat{Q}_t \ ,$$

which leads to perfect stabilization of the asset price gap, $\hat{Q}_t = 0$ for all t, as the unique rational expectations equilibrium of the economy outside the ZLB.[14] Section 3 discusses the stabilization and uniqueness properties of the model with a binding ZLB. Section 4 considers different forward guidance strategies that deviate from equation (11) by temporarily committing to a distinct set of passive policy rules (Odyssean guidance), whose stabilization and uniqueness properties are further discussed later.

[12] A conventional definition using the output gap leads to a comparable expression in our model, since both variables are proportional in equilibrium.

[13] In addition to the Taylor principle $\phi_q > 0$, Lee and Dordal i Carreras (2023) establish that targeting the risk-adjusted natural rate or its risk-premium component is an additional necessary condition for ensuring equilibrium uniqueness in models incorporating higher-order terms in the dynamic IS equation.

[14] See Blanchard and Kahn (1980) and Buiter (1984) for a detailed presentation of the necessary conditions required for this uniqueness result.

3 The Zero Lower Bound

ZLB Recession Following the approach of Werning (2012), we consider a scenario where the interest rate is brought to the ZLB by a deterministic shift in the natural rate of interest, r_t^n. To that purpose, we consider the case where $\sigma_t = \bar{\sigma}$ for $0 \leq t \leq T$ and $\sigma_t = \underline{\sigma} < \bar{\sigma}$ for $t \geq T$. More specifically, we assume that TFP volatilites during these periods are such that the natural rate r_t^n satisfies: $\underline{r} \equiv r^n(\bar{\sigma}) = \rho + g - \bar{\sigma}^2 < 0$ and $\bar{r} \equiv r^n(\underline{\sigma}) = \rho + g - \underline{\sigma}^2 > 0$, resulting in the ZLB binding in the first period. The case where the ZLB duration T is stochastic can be similarly analyzed, and illustrated in Online Appendix E.

Recovery Without Guidance We begin our study of ZLB recessions by examining the benchmark scenario: economic recovery in the absence of forward guidance or other macroprudential policies. After period T, we assume that the monetary authority follows the Taylor rule presented in equation (11), achieving perfect economic stabilization defined by $\hat{Q}_t = 0$ for $t \geq T$. We infer by backward induction from equation (10) that perfect stabilization with certainty at T necessarily implies the absence of volatility in the asset price gap $\hat{Q}_t$ process in the preceding periods, $t < T$.[15] Therefore, it follows that $\sigma_t^q = 0$ and $r_t^T = \underline{r} < 0$ for $t < T$ whenever the monetary authority can credibly commit to follow the Taylor rule in equation (11) for $t \geq T$. In this scenario, the dynamics of $\hat{Q}_t$ according to (10) simplify to:

$$d\hat{Q}_t = -\underline{r}\, dt \ , \qquad \text{for } t < T \ , \tag{12}$$

with associated boundary condition $\hat{Q}_T = 0$ and initial asset price gap given by $Q_0 = \underline{r}\,T$. The trajectory of $\{\hat{Q}_t\}$ following equation (12) is illustrated in Figure 1.

The initial increase in σ_t from $\underline{\sigma}$ to $\bar{\sigma}$ raises the risk premium from $\text{rp}_2^n = (\underline{\sigma})^2$ to $\text{rp}_1^n = \bar{\sigma}^2$. This leads to a decline in asset prices $\hat{Q}_t$ because the ZLB prevents the risk-free rate from falling into negative territory, as would be necessary for complete stabilization. As a result, there is a diminished appetite among capitalists for stock market investments, leading to a reduction in both aggregate financial wealth and consumption demand.[16] This

[15] For instance, at $T - \Delta$, where Δ is an infinitesimally small time interval, $\sigma_{T-\Delta}^q = 0$ is the only rational solution to equation (10) consistent with $\hat{Q}_T = 0$ for any possible realization of the stochastic component of the TFP process, $dZ_{T-\Delta}$. This result deterministically pins down the asset price gap of the preceding period, $\hat{Q}_{T-\Delta}$, leading by backward induction to $\sigma_t^q = 0$ for $t \leq T$.

[16] While Caballero and Farhi (2017) demonstrate that an increased demand for safe assets can drive the economy into recession under ZLB constraints, our analysis suggests that it encourages investors to withdraw

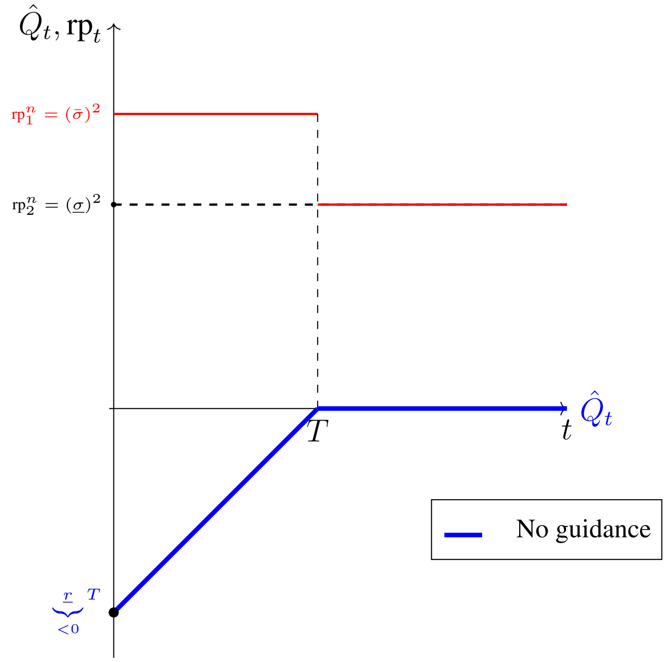

Figure 1: ZLB dynamics, economic recovery without guidance (Benchmark).

path is consistent with the dynamics described in Werning (2012) and Cochrane (2017), despite our model featuring a distinct IS equation (10) with endogenous volatility σ_t^q influencing the drift in the $\hat{Q}_t$ process, a departure from traditional New-Keynesian models. This result arises because ensuring future stabilization for $t \geq T$ effectively eliminates any excess endogenous volatility σ_t^q during a ZLB episode.

Remarks Central banks can prevent the emergence of endogenous volatility σ_t^q at the ZLB through a 'credible' commitment to stabilize the business cycle by a predetermined future date $T < +\infty$. Even if the monetary authority is constrained by the ZLB and thus unable to adhere to the policy rule outlined in (10), which directly targets the risk-premium, the additional financial stability costs resulting from policy inaction can be effectively managed, or even completely eliminated, by pledging to stabilize upon exiting the ZLB. One implication of this result is that the impact of the ZLB could vary significantly between countries: those with monetary authorities committed to stabilization after the ZLB period may only face the demand-driven recession described in this Section. In contrast, countries lacking the capacity or willingness to stabilize in the future might incur additional costs

their wealth from the stock market, thus reducing stock market value and aggregate demand, akin to the findings of Caballero and Simsek (2020).

due to potential increases in σ_t^q during a ZLB episode. Exploration of these scenarios is left for future research.

Additionally, we can relax the assumption that T is deterministic. Even if T is stochastic following a distribution, our result that $\sigma_t^q = \sigma_t^{q,n} = 0$ before T if the monetary authority perfectly closes the gap after T still carries on. It is because the consumption process during the ZLB would be dependent only on T-realizations. We illustrate this extension in Online Appendix E, and therefore, keep focusing on the simplest case where T is deterministic.

4 Forward Guidance

This section analyzes two different forward guidance strategies and explores the potential stabilization trade-offs involved in the use of these policy tools.

4.1 Traditional Forward Guidance

We define traditional forward guidance as the communication strategy where the central bank credibly commits to maintaining a zero policy rate for a duration of time $\hat{T}^{\text{TFG}} > T$ exceeding the initial period of high fundamental volatility. We further assume that the central bank reverts to the policy rule defined in equation (11) after the forward guidance period ends, resulting in a perfect stabilization of both the business cycle and financial markets for $t \geq \hat{T}^{\text{TFG}}$. Following from the backward induction rationale presented in Section 3, stabilization with certainty after $\hat{T}^{\text{TFG}}$ results in the absence of endogenous financial volatility, $\sigma_t^q = 0$, for $t < \hat{T}^{\text{TFG}}$. The dynamics of $\hat{Q}_t$ are described by

$$d\hat{Q}_t = \begin{cases} -\underline{r}\ dt\ , & \text{for } t < T\ , \\ -\bar{r}\ dt\ , & \text{for } T \leq t < \hat{T}^{\text{TFG}}\ , \end{cases} \tag{13}$$

with associated boundary condition $\hat{Q}_{\hat{T}^{\text{TFG}}} = 0$, resulting in an initial asset price gap of $\hat{Q}_0 = \underline{r}\, T + \bar{r}\, (\hat{T}^{\text{TFG}} - T)$.

The dynamics of $\{\hat{Q}_t\}$ governed by equation (13) are depicted in Figure 2. Traditional forward guidance induces an artificial economic boom between T and $\hat{T}^{\text{TFG}}$, thereby alleviating recessionary pressures within the interval $0 \leq t < T$. Specifically, traditional forward guidance increases asset prices between T and $\hat{T}^{\text{TFG}}$, which results in a narrower initial asset price gap $\hat{Q}_0$ due to the forward-looking nature of stock markets.

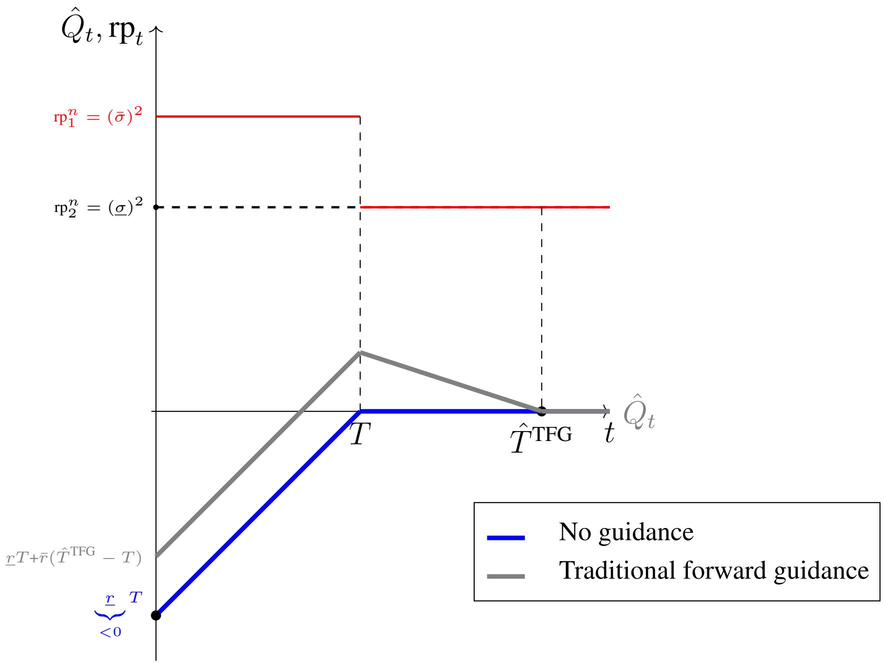

Figure 2: ZLB dynamics under traditional forward guidance.

Optimal Traditional Forward Guidance To determine the optimal forward guidance duration $\hat{T}^{\mathrm{TFG}}$, we minimize the quadratic loss function represented by:

$$\mathbb{L}^Q \left(\{\hat{Q}_t\}_{t \geq 0} \right) = \mathbb{E}_0 \int_0^\infty e^{-\rho t} \hat{Q}_t^2 \; dt \; , \tag{14}$$

subject to the dynamics outlined in equation (13). The first-order condition with respect to $\hat{T}^{\mathrm{TFG}}$ results in:

$$\int_0^\infty e^{-\rho t} \hat{Q}_t dt = 0 \; . \tag{15}$$

Section 4.4 presents a summary of the principal statistics and welfare gains resulting from the adoption of the optimal traditional forward guidance policy outlined in this discussion.

In the next section, we argue that central banks might voluntarily forgo perfect stabilization in the future to reduce financial volatility at the ZLB and potentially achieve higher welfare than with the traditional forward guidance policy described here. We term this approach a 'higher-order' forward guidance policy.

4.2 Higher-Order Forward Guidance

The principal cause of ZLB recessions in our model is an excessively high risk premium, driven by increased fundamental volatility σ_t. As a result, central banks might alternatively consider focusing on mitigating financial risk by steering agents' actions toward a favorable trajectory for the asset price volatility $\{\sigma_t^q\}$ during the ZLB period, aiming to support asset prices and consumption demand.[17]

Context In the traditional forward guidance policy previously discussed, the central bank's commitment to perfect stabilization (with certainty) at $\hat{T}^{\text{TFG}}$ facilitates a smoother transition toward economic recovery. However, this approach prevents any deviation of σ_t^q from zero, its natural level, during the ZLB period, as depicted in Figure 3. This suggests that to sustain alternative equilibria where σ_t^q deviates from zero, the central bank must refrain from promising perfect stabilization upon exiting the ZLB at $\hat{T}^{\text{TFG}}$, as illustrated in Figure 4.

1. Central bank achieves perfect stabilization with certainty after $\hat{T}^{\text{TFG}}$ (i.e., $\hat{Q}_t = 0$, for $t \geq \hat{T}^{\text{TFG}}$)

$\downarrow$

2. $\hat{Q}_{\hat{T}^{\text{TFG}}} = 0$ guarantees $\sigma_t^q = \sigma_t^{q,n} = 0$, $\text{rp}_t = \text{rp}_t^n$ for $t < \hat{T}^{\text{TFG}}$

Figure 3: Mechanism under traditional forward guidance.

$\neg$2. $\sigma_t^q < \sigma_t^{q,n} = 0$, $\text{rp}_t < \text{rp}_t^n$ for $t < \hat{T}^{\text{TFG}}$

$\downarrow$

$\neg$1. $\hat{Q}_{\hat{T}^{\text{TFG}}} \neq 0$: central bank commits not to perfectly stabilize the economy after $\hat{T}^{\text{TFG}}$

Figure 4: Mechanism under higher-order forward guidance.

Implementation We define $\hat{T}^{\text{HOFG}}$ as the duration of zero policy rate under our 'higher-order' policy. We model the commitment constraint described in Figure 4 by assuming that after the forward guidance regime with i_t equal to zero ends at $\hat{T}^{\text{HOFG}}$, the monetary

[17] The risk premium, rp_t, is given by $\text{rp}_t = (\bar{\sigma} + \sigma_t^q)^2$ for $t < T$ and $\text{rp}_t = (\underline{\sigma} + \sigma_t^q)^2$ for $T \leq t < \hat{T}^{\text{TFG}}$. Therefore, a negative σ_t^q can reduce the risk premium below its natural level, thereby improving asset prices and aggregate demand at the ZLB.

authority implements a *passive policy rule* with i_t fixed at $\bar{r}$, which allows for the existence of multiple equilibria. The central bank then coordinates the economy's agents into an optimal path within the admissible solutions set, subject to the constraints: $\sigma_t^q = 0$ for $t \geq \hat{T}^{\text{HOFG}}$ and $\mathbb{E}_0 \hat{Q}_\infty = 0$. The latter is necessary to meet the economy's transversality condition, while the former simplifies the optimization problem by assuming the central bank ends its influence on financial market volatility at the conclusion of the forward guidance period. Together with the dynamic IS equation in (10), these constraints indicate that the asset price gap is initially expected to close, $\mathbb{E}_0 \hat{Q}_{\hat{T}^{\text{HOFG}}} = 0$, by the end of the forward guidance period at $\hat{T}^{\text{HOFG}}$. In Section 4.3, we will additionally assume that the central bank permanently reverts to the active Taylor rule in equation (11) with a constant probability less than one after $\hat{T}^{\text{HOFG}}$.

Formalism We denote the natural risk premiums as $\text{rp}_1^n \equiv \bar{\sigma}^2$ for $t < T$ (high fundamental volatility region), $\text{rp}_2^n \equiv \underline{\sigma}^2$ for $T \leq t < \hat{T}^{\text{HOFG}}$ (low fundamental volatility region), and $\text{rp}_3^n \equiv \underline{\sigma}^2$ for $t \geq \hat{T}^{\text{HOFG}}$ (low fundamental volatility region post-forward guidance period).[18]

$$\boxed{\neg 2.\ \sigma_t^q = \sigma_1^{q,L} < 0 \text{ for } t < T;\ \sigma_2^{q,L} < 0 \text{ for } T \leq t \leq \hat{T}^{\text{HOFG}};\ \sigma_t^{q,n} = 0 \text{ for } t > \hat{T}^{\text{HOFG}}}$$
$$\downarrow$$
$$\boxed{\neg 1.\ \hat{Q}_{\hat{T}^{\text{HOFG}}} \neq 0\text{: central bank pegs its policy rate } i_t = \bar{r} \text{ after } \hat{T}^{\text{HOFG}}}$$

Figure 5: Simplified higher-order forward guidance.

We can simplify the optimization problem by assuming that the central bank maintains consistent financial volatility and risk-premium levels within each regime. Specifically, financial volatility σ_t^q is set to be $\sigma_1^{q,L}$ for $t < T$, $\sigma_2^{q,L}$ for $T \leq t < \hat{T}^{\text{HOFG}}$, and zero for $t \geq \hat{T}^{\text{HOFG}}$. The risk-premia associated with each period are $\text{rp}_1 \equiv (\bar{\sigma} + \sigma_1^{q,L})^2 < \text{rp}_1^n$ for $t < T$, $\text{rp}_2 \equiv (\underline{\sigma} + \sigma_2^{q,L})^2 < \text{rp}_2^n$ for $T \leq t < \hat{T}^{\text{HOFG}}$, and $\text{rp}_3 \equiv (\underline{\sigma})^2$ for $t \geq \hat{T}^{\text{HOFG}}$.[19] This simplified problem is represented in Figure 5. Finally, the risk-adjusted natural rate in equation (9) is expressed as r_1^T for $t < T$ and r_2^T for $T \leq t < \hat{T}^{\text{HOFG}}$, each being a function

[18] Risk premium is defined as $\text{rp}_t = (\sigma_t + \sigma_t^q)^2$, and the expression for the natural level stems from the existence of zero endogenous financial volatility in a flexible price economy, where $\sigma_t^{q,n} = 0$ for all t.

[19] Proposition 2 later proves that $\sigma_1^{q,L} < 0$ and $\sigma_2^{q,L} < 0$ at the optimum. For illustration purposes, we assume these conditions are satisfied in the rest of the argument of this section.

of $\sigma_1^{q,L}$ and $\sigma_2^{q,L}$, respectively. This is represented by:

$$
\begin{aligned}
r_1^T\left(\sigma_1^{q,L}\right) &\equiv \rho + g - \frac{\bar{\sigma}^2}{2} - \frac{\left(\bar{\sigma} + \sigma_1^{q,L}\right)^2}{2} > \underline{r} \equiv r_1^T(0) \;\text{ when }\; \sigma_1^{q,L} < 0 \;, \\
r_2^T\left(\sigma_2^{q,L}\right) &\equiv \rho + g - \frac{\underline{\sigma}^2}{2} - \frac{\left(\underline{\sigma} + \sigma_2^{q,L}\right)^2}{2} > \bar{r} \equiv r_2^T(0) \;\text{ when }\; \sigma_2^{q,L} < 0 \;.
\end{aligned}
\tag{16}
$$

From equation (16), we observe that lower risk premia during the forward guidance period up to $\hat{T}^{\text{HOFG}}$ lead to increased risk-adjusted rates and, consequently, higher values of the asset price gap $\{\hat{Q}_t\}$ along the expected equilibrium path (in comparison to a traditional forward guidance policy of the same duration). This results in reduction of the expected quadratic loss function in (14). However, as indicated by our IS equation (10), a σ_t^q different from zero introduces stochastic fluctuations in the trajectory of $\hat{Q}_t$, resulting in potential additional stabilization costs in the future. The green line in Figure 6 illustrates the expected trajectory (or deterministic component) of $\{\hat{Q}_t\}$ under a higher-order forward guidance policy as detailed in this section. The dashed lines alongside the expected path depict two possible sample paths that stem from stochastic variations in $\{\hat{Q}_t\}$.

In summary, central banks operating under our higher-order guidance with commitment face a trade-off between achieving lower risk premiums and higher asset price levels prior to $\hat{T}^{\text{HOFG}}$, and the subsequent costs of de-stabilization. This balancing act involves a careful choice of $\sigma_1^{q,L}$, $\sigma_2^{q,L}$, and $\hat{T}^{\text{HOFG}}$, as we discuss next. It will turn out that due to the additional stabilization effects coming from negative $\sigma_1^{q,L}$ and $\sigma_2^{q,L}$, the duration of zero policy rate $\hat{T}^{\text{HOFG}}$ falls from $\hat{T}^{\text{TFG}}$.

Optimal Higher-Order Forward Guidance The initial asset price gap $\hat{Q}_0$ is determined by the condition $\mathbb{E}_0\hat{Q}_{\hat{T}^{\text{HOFG}}} = 0$ previously discussed and the dynamic IS equation in (10) as follows:

$$
\hat{Q}_0 = r_1^T(\sigma_1^{q,L})\,T + r_2^T(\sigma_2^{q,L})\,(\hat{T}^{\text{HOFG}} - T)\;. \tag{17}
$$

The central bank minimizes the loss function given by (14) by selecting the optimal values

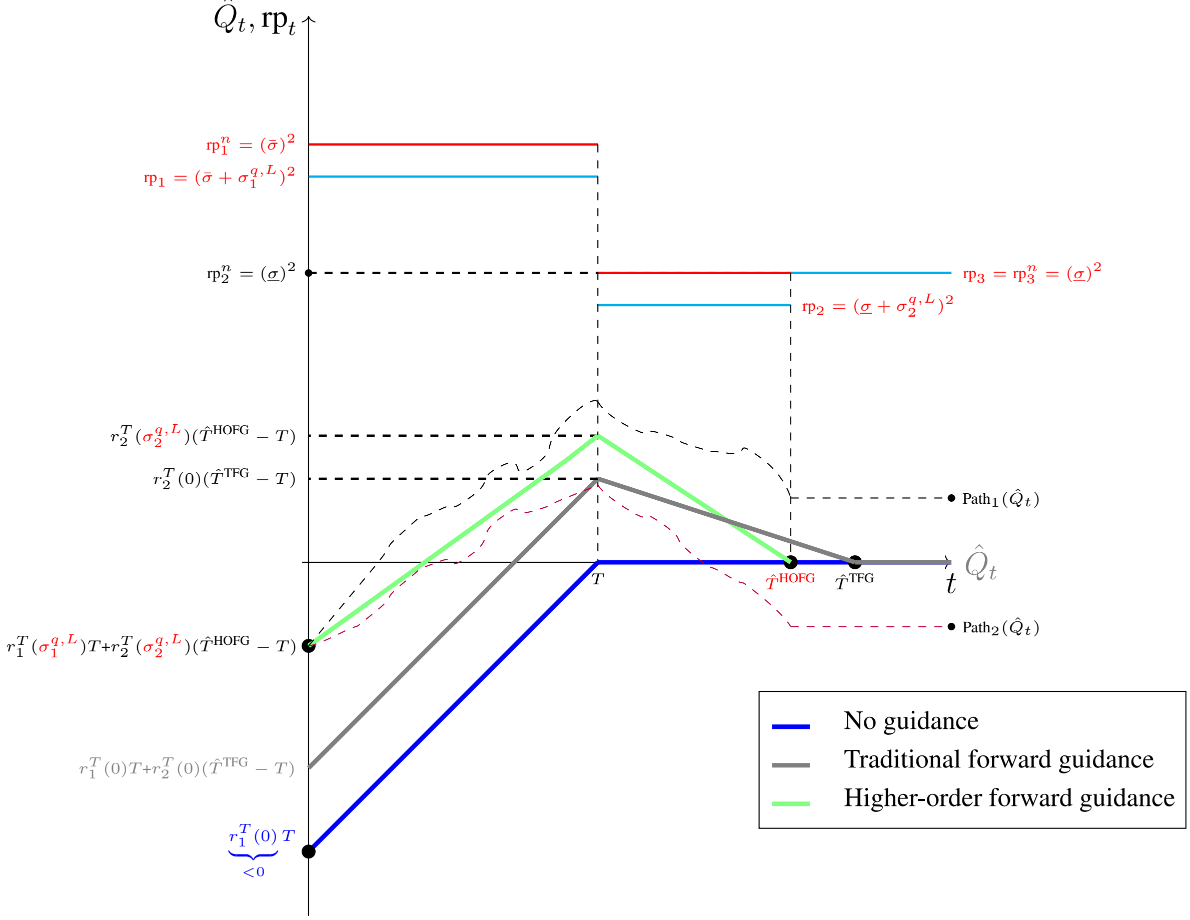

Figure 6: Intervention dynamics of $\{\hat{Q}_t\}$ with $\sigma_1^{q,L} < 0$, $\sigma_2^{q,L} < 0$, and $\hat{T}^{\mathrm{HOFG}} < \hat{T}^{\mathrm{TFG}}$.

for $\sigma_1^{q,L}$, $\sigma_2^{q,L}$, and $\hat{T}^{\mathrm{HOFG}}$. The formulation of the optimization problem is:

$$\min_{\sigma_1^{q,L}, \sigma_2^{q,L}, \hat{T}^{\mathrm{HOFG}}} \mathbb{E}_0 \int_0^\infty e^{-\rho t} \hat{Q}_t^2 \, dt, \text{ s.t. } d\hat{Q}_t = \begin{cases} -r_1^T(\sigma_1^{q,L})dt + \sigma_1^{q,L} dZ_t, & \text{for } t < T, \\ -r_2^T(\sigma_2^{q,L})dt + \sigma_2^{q,L} dZ_t, & \text{for } T \leq t < \hat{T}^{\mathrm{HOFG}}, \\ 0, & \text{for } t \geq \hat{T}^{\mathrm{HOFG}}, \end{cases} \tag{18}$$

with $\hat{Q}_0$ determined by equation (17). The following Proposition 2 summarizes the resulting optimal commitment path for the central bank under higher-order forward guidance.

Proposition 2 (Optimal Commitment Path) *The solution to the central bank's higher-order forward guidance optimization problem in (18) results in an optimal commitment path characterized by $\sigma_1^{q,L} < 0$, $\sigma_2^{q,L} < 0$, and $\hat{T}^{HOFG} < \hat{T}^{TFG}$. In addition, optimal higher-order forward guidance always results in an equal or lower expected quadratic loss than* *the traditional forward guidance discussed in Section 4.1.*

Proof. See Appendix II. The latter part follows from the fact that when $(\sigma_1^{q,L}, \sigma_2^{q,L}, \hat{T}^{\text{HOFG}}) = (0, 0, \hat{T}^{\text{TFG}})$, the trajectory of the asset price gap $\{\hat{Q}_t\}$ becomes identical to that of a traditional forward guidance policy with duration $\hat{T}^{\text{TFG}}$. Thus, an optimal choice of these parameters will always lead to an equal or lower value of the quadratic loss function presented in equation (14).

■

4.3 Higher Order Forward Guidance *with* Stochastic Stabilization

In the previous section, we assumed that following the end of the forward guidance regime at $\hat{T}^{\text{HOFG}}$, the monetary authority would passively peg the policy rate i_t to the natural rate $\bar{r}$ and set σ_t^q to zero indefinitely. This setup allows for σ_t^q to deviate from zero during the ZLB period, as illustrated in Figure 6. Moving to this section, we relax these assumptions while maintaining the support for the existence of multiple equilibria provided by the earlier framework. Now, we assume that after forward guidance ends, the central bank not only follows the outlined passive rule but also commits to a stochastic return to the perfect stabilization rule in equation (11). This commitment is represented as a constant probability outcome determined by a Poisson process. Accordingly, the process for the asset price gap $\hat{Q}_t$ after $\hat{T}^{\text{HOFG}}$ becomes:

$$d\hat{Q}_t = -\hat{Q}_t d\Pi_t \, , \quad \text{s.t.} \;\; d\Pi_t = \begin{cases} 1 \, , & \text{with probability } \nu dt \, , \\ 0 \, , & \text{with probability } 1 - \nu dt \, , \end{cases}$$

where $d\Pi_t$ is a Poisson random variable, with rate parameter $\nu \geq 0$.[20] The central bank's optimization problem can be expressed as:

$$\min_{\sigma_1^{q,L}, \sigma_2^{q,L}, \hat{T}^{\text{HOFG}}} \mathbb{E}_0 \int_0^{\hat{T}^{\text{HOFG}}} e^{-\rho t} \hat{Q}_t^2 \, dt + \int_{\hat{T}^{\text{HOFG}}}^{\infty} e^{-\rho t} \cdot e^{-\nu\left(t - \hat{T}^{\text{HOFG}}\right)} \cdot \hat{Q}_t^2 \, dt \, ,$$
$$\text{s.t.} \;\; d\hat{Q}_t = \begin{cases} -r_1^T(\sigma_1^{q,L})dt + \sigma_1^{q,L} dZ_t, & \text{for } t < T, \\ -r_2^T(\sigma_2^{q,L})dt + \sigma_2^{q,L} dZ_t, & \text{for } T \leq t < \hat{T}^{\text{HOFG}}, \\ 0, & \text{for } t \geq \hat{T}^{\text{HOFG}}, \end{cases} \tag{19}$$

[20] Here, ν is treated as an exogenous parameter determined by external factors. If the central bank could choose an optimal ν, it would select $\nu \to +\infty$, as demonstrated in Online Appendix D.

with $\hat{Q}_0$ determined by equation (17). Proposition 3 outlines the optimal commitment path for the central bank under higher-order forward guidance with stochastic stabilization.

Proposition 3 (Optimal Commitment Path with Stochastic Stabilization) *The solution to the central bank's forward guidance optimization problem in (19) results in an optimal commitment path characterized by $\sigma_1^{q,L} < 0$, $\sigma_2^{q,L} < 0$, and $\hat{T}^{HOFG} < \hat{T}^{TFG}$. In addition, optimal higher-order forward guidance with a stochastic stabilization probability always results in an equal or lower expected quadratic loss than the traditional forward guidance discussed in Section 4.1.*

Furthermore, an increased probability of stabilization, indicated by higher values of ν, leads to a reduction in the optimal values of $\sigma_1^{q,L}$ and $\sigma_2^{q,L}$, resulting in a decrease in risk premia at the ZLB.

Proof. See Online Appendix D. The first part of the proposition directly extends the results of Proposition 2 to a stochastic stabilization environment. The latter part of the proposition is based on the reduced costs of a more aggressive countercyclical policy at the ZLB when future stabilization is more likely.

■

Finally, Corollary 1 asserts that introducing a minimal degree of uncertainty about the timing of future stabilization in its communications is always optimal for the central bank. This approach facilitates the application of higher-order forward guidance, resulting in equilibrium paths that are strictly superior from a quadratic loss perspective, compared to those under traditional forward guidance.

Corollary 1 (Discontinuity at the Limit) *The limit case where stabilization parameter ν equals $+\infty$ corresponds to the traditional forward guidance problem described in Section 4.1. As ν approaches $+\infty$ from the left, the central bank's expected quadratic loss function exhibits a discontinuity. Specifically, the expected quadratic loss is always lower when there's a minimal probability of stabilization. Formally:*

$$\lim_{\nu \to +\infty^-} \mathbb{L}^{Q,*}\left(\{\hat{Q}_t\}_{t\geq 0}, \nu\right) < \mathbb{L}^{Q,*}\left(\{\hat{Q}_t\}_{t\geq 0}, \nu = \infty\right) ,$$

where $\mathbb{L}^*\left(\{\hat{Q}_t\}_{t\geq 0}, \nu\right)$ represents the quadratic loss function defined in equation (14), evaluated at its optimum for an economy characterized by a Poisson rate ν.

Proof. See Online Appendix D. The intuition behind the statement's first part is that the probability of immediate stabilization upon exiting the forward guidance period at $\hat{T}^{\text{HOFG}}$ becomes one when $\nu = +\infty$, aligning with the scenario of the traditional forward guidance policy in Section 4.1. The second part is based on that higher-order guidance consistently results in an equal or lower expected quadratic loss compared to the traditional guidance, regardless of ν, as outlined in Proposition 3.

■

4.4 Welfare Comparison

For the quantitative evaluation of different forward guidance policies discussed in this paper, we simulate optimal commitment paths at the ZLB under three scenarios: (i) no forward guidance, (ii) traditional forward guidance, and (iii) higher-order forward guidance with varying probabilities of stabilization. The initial ZLB duration T is set at 20 quarters to reflect the lengthy ZLB periods that followed the global financial crisis. The Poisson rate parameter ν in the higher-order forward guidance policy is first calibrated to zero, denoting a zero probability of reverting to an active policy rule, and then to one, signifying the expectation of resuming an active policy rule one quarter after the forward guidance period concludes. The remaining model parameters are calibrated based on values commonly found in the literature, as detailed in Appendix Table I.1.

We define the loss function $\mathbb{L}$ as the quadratic output loss per quarter, and approximate it by:

$$\mathbb{L}^{Y}_{\text{Per-period}} \equiv \rho \int_0^\infty e^{-\rho t} \mathbb{E}_0 \left(\hat{Y}_t^2 \right) \approx \zeta^2 \cdot \rho \int_0^\infty e^{-\rho t} \frac{1}{s} \sum_{i=1}^{s} \left(\hat{Q}_t^{(i)} \right)^2 dt \ ,$$

where $\zeta > 0$ follows from the relationship $\hat{Y}_t = \zeta \hat{Q}_t$, as derived in equation (B.1) of Online Appendix B. Here, $\hat{Q}_t^{(i)}$ represents the i^{th} simulated stochastic sample path of the asset price gap.[21] We consider a scenario characterized by a one-time ZLB recession commencing in period zero, without any expectation of future recurrence. Therefore, $\mathbb{L}$ is to be interpreted as the expected conditional loss associated with a single ZLB episode.

Table 1 presents the results of our simulation, where $\sigma_1^{q,L}$ and $\sigma_2^{q,L}$ are expressed as percentages of the fundamental volatilities $\bar{\sigma}$ and $\underline{\sigma}$, respectively. The initial columns report the

[21] We use $s = 10^4$ randomly simulated sample paths to approximate the quadratic loss of the higher-order forward guidance policies.

Policy	**No guidance**	**Traditional**	**Higher-Order** (no stochastic stabilization)	**Higher-Order** (*with* stoch. stab., $\nu = 1$)
$\sigma_1^{q,L}$	0	0	-1.27%	-4.13%
$\sigma_2^{q,L}$	0	0	-0.24%	-3.79%
$\hat{T}^{\text{HOFG}}$	20	25.27	25.09	24.68
$\mathbb{L}^{Y}_{\text{Per-period}}$	7%	1.93%	1.81%	1.69%

Table 1: Policy comparisons.

effectiveness of traditional guidance, showing the central bank extending the ZLB for just over a year, reducing total loss by about five percentage points. These findings are aligned with existing literature (see Campbell et al. (2012, 2019), Del Negro et al. (2013), McKay et al. (2016)).[22] The last two columns provide summary statistics on optimal higher-order guidance implementation under the two stabilization regimes discussed above. The results are consistent with higher-order guidance characteristics described in Propositions 2 and 3. Higher-order guidance, compared to traditional policy, further reduces ZLB costs by a moderate 0.12%-0.24% per quarter through lower financial market volatility during the guidance period, and allows for an earlier exit from the ZLB. Finally, the last column reports that gains from higher-order guidance double when there is a positive probability of returning to full stabilization in the future.

In Section 5, we shift our focus to explore potential macroprudential interventions from a fiscal perspective, aimed at increasing asset prices $\hat{Q}_t$ and stabilizing the business cycle during a ZLB recession.

5 Macroprudential Policies

This section examines two types of macroprudential policies designed to stimulate the economy at the ZLB. Firstly, we consider a fiscal subsidy aimed at encouraging capitalists to undertake higher levels of risk, thereby boosting asset prices and other real economic activities. Secondly, we explore the impact of direct fiscal transfers from capitalists to hand-to-mouth workers, who typically exhibit a higher marginal propensity to consume. This policy is shown to increase overall stock market dividends, and consequently, asset

[22] These studies also note the issue of traditional forward guidance being overly potent in plain vanilla New-Keynesian frameworks compared to empirical estimates. This paper does not include the quantitative adjustments proposed in the literature to address this discrepancy, focusing instead on the distinctions between traditional and higher-order forward guidance policies.

prices $\hat{Q}_t$ and consumption. To assess the impact of macroprudential policies on the business cycle, forward guidance is excluded from our analysis in this section. We maintain the same scenario as outlined in Section 3, and assume that monetary policy reverts to the perfect stabilization rule specified in equation (11) for $t \geq T$.

5.1 Fiscal Subsidy on Stock Market Investment

In the period up to T, where $r_t^n = \underline{r} < 0$ and monetary policy is constrained by the ZLB, the risk-premium level $\mathrm{rp}_1^n = \bar{\sigma}^2$ required by capitalists leads to a reduction in asset prices, $\hat{Q}_t$. To counteract this, we propose a subsidy policy aimed at incentivizing capitalists' holdings of the risky stock market index. This intervention is expected to increase $\hat{Q}_t$, thereby addressing the aggregate demand externalities responsible for dragging the economy into a ZLB recession.[23]

We begin by examining a government subsidy for the purchase of (risky) stock market index shares.[24] Specifically, instead of the usual expected return i_t^m, a capitalist earns an expected return of $(1+\tau)i_t^m$ for every dollar invested in the stock market, where $\tau \geq 0$ is the stock subsidy. To fund this intervention, the government imposes a 'lump-sum' tax L_t on capitalists. Consequently, a capitalist solves the optimization problem with a modified flow budget constraint given by:

$$\begin{aligned} &\max_{C_t,\theta_t} \mathbb{E}_0 \int_0^\infty e^{-\rho t} \log C_t \, dt \\ &\text{s.t. } da_t = \left(a_t \left(i_t + \theta_t((1+\tau)i_t^m - i_t)\right) - \bar{p}C_t - L_t\right) dt + \theta_t a_t \left(\bar{\sigma} + \sigma_t^q\right) dZ_t \, . \end{aligned} \tag{20}$$

In equilibrium, capitalists finance the stock market subsidy by paying taxes L_t equal to $\tau \bar{p} A_t Q_t i_t^m$. Setting $\theta_t = 1$ in equilibrium, we can express the stock market's expected return as follows:

$$i_t^m = \frac{i_t + (\bar{\sigma} + \sigma_t^q)^2}{1+\tau} = \underbrace{\rho}_{\substack{\text{Dividend} \\ \text{yield}}} + \underbrace{g + \mu_t^q + \sigma_t \sigma_t^q}_{\text{Capital gain}} \, . \tag{21}$$

[23] Numerous studies have examined the link between externalities (e.g., pecuniary or aggregate-demand) and macroprudential policies. Notable references include Caballero and Krishnamurthy (2001), Lorenzoni (2008), Farhi et al. (2009), Bianchi and Mendoza (2010), Jeanne and Korinek (2010), Stein (2012), Farhi and Werning (2012, 2016, 2017), Korinek and Simsek (2016), Dàvila and Korinek (2018), among others.

[24] In our model, a subsidy for stock investments functions similarly to a tax break on capital income, a policy commonly implemented *in practice* by governments. We opt for the subsidy model for simplicity in notation.

As detailed in Section 3, given that σ_t^q and i_t equal zero for $t \leq T$, equation (21) simplifies to

$$i_t^m = \frac{\bar{\sigma}^2}{1+\tau} \ ,$$

which is lower than $\bar{\sigma}^2$ and inversely proportional to τ. Thus, a positive subsidy rate $\tau > 0$ increases $\hat{Q}_t$ along the path up to time T, when the economy achieves perfect stabilization with $\hat{Q}_T = 0$. Proposition 4 summarizes this result.

Proposition 4 (Fiscal Subsidy on Stock Market Expected Returns) *Under the ZLB environment of Section 3, where a fiscal subsidy $\tau \geq 0$ is applied to the expected return of stock markets, the dynamics of $\hat{Q}_t$ during the period $t < T$ are given by:*

$$d\hat{Q}_t = -\left(\underbrace{\underline{r}}_{\equiv r^n(\bar{\sigma})<0} + \underbrace{\frac{\tau}{1+\tau}\bar{\sigma}^2}_{>0} \right) dt \ , \tag{22}$$

for $\underline{r} + \frac{\tau}{1+\tau}\bar{\sigma}^2 < 0$ and $\hat{Q}_T = 0$. When $\underline{r} + \frac{\tau}{1+\tau}\bar{\sigma}^2 > 0$, the subsidy $\tau > 0$ lifts the economy out of the ZLB and immediate stabilization becomes possible by adhering to the policy rule outlined in equation (11)*.*

Proof. See Appendix II.

■

In equation (22), a positive subsidy $\tau > 0$ increases the effective natural rate from $\underline{r}$ to $\underline{r} + \frac{\tau}{1+\tau}\bar{\sigma}^2$. This rise narrows the gap between the ZLB and the 'effective' natural rate, consequently raising $\hat{Q}_t$ relative to the scenario described in Section 3. It is important to note that as τ approaches infinity, the expression $\underline{r}+\frac{\tau}{1+\tau}\bar{\sigma}^2$ converges to $\underline{r}+\bar{\sigma}^2 = \rho+g > 0$. In this situation, the economy moves away from the ZLB and the monetary authority can achieve perfect stabilization by adhering to the policy rule outlined in equation (11).

Tax on whom? We now consider an alternative funding scheme for the stock market subsidy τ by imposing a lump-sum tax L_t on hand-to-mouth workers. Under this policy, the budget constraint of the workers (1) becomes

$$\frac{w_t}{\bar{p}} N_{W,t} = C_{W,t} + \frac{L_t}{\bar{p}} \ . \tag{23}$$

Hand-to-mouth workers, characterized by a marginal propensity to consume of one, experience a proportional reduction in their consumption due to taxation. This fall in workers' consumption adversely impacts stock dividends and prices, $\hat{Q}_t$. In this context, the formula for the stock market's expected return i_t^m is as follows:

$$i_t^m = \underbrace{\frac{y_t - \frac{w_t}{\bar{p}} N_{W,t}}{A_t Q_t}}_{\text{Dividend yield}} + \mathbb{E}_t \left[\frac{d(\bar{p} A_t Q_t)}{\bar{p} A_t Q_t} \frac{1}{dt} \right] = \underbrace{\rho - \tau i_t^m}_{\text{Dividend yield}} + \mathbb{E}_t \left[\frac{d(\bar{p} A_t Q_t)}{\bar{p} A_t Q_t} \frac{1}{dt} \right], \tag{24}$$

where we used an equilibrium tax equal to $\tau i_t^m \bar{p} A_t Q_t$ to obtain the last equality. Proposition 5 summarizes our findings, highlighting the crucial role of tax scheme design in determining the effectiveness of the macroprudential policy.

Proposition 5 (Fiscal Subsidy and Tax on Workers) *The positive impact of a subsidy τ on asset prices is precisely offset by the reduced consumption of hand-to-mouth workers due to taxation L_t. Consequently, this results in no net effect on the dynamics of $\{\hat{Q}_t\}$ during a ZLB episode, apart from a redistribution of wealth from workers to capitalists. The trajectory of asset prices under this taxation scheme corresponds with the benchmark scenario, which lacks forward guidance and macroprudential interventions, as depicted in Figure 1.*

Proof. See Appendix II.

■

5.2 Fiscal Redistribution

Lastly, we consider a redistribution policy in the form of a fiscal transfer $L_t > 0$ from capitalists to hand-to-mouth workers during a ZLB episode.[25] This policy increases aggregate demand due to the high marginal propensity to consume of workers and, in turn, the total dividends paid by the stock market index. The expected return on the stock market i_t^m then

[25] A policy subsidizing firms' payroll, financed through a lump-sum tax L_t on capitalists, produces identical results. When firms incur net payroll costs of $w_t N_{W,t} - L_t$, the consequent rise in employment effectively creates a transfer of income equivalent to L_t to the workers. We opt for the direct transfer formulation for simplicity in notation.

becomes:

$$i_t^m = \frac{y_t - \frac{w_t}{\bar{p}} N_{W,t}}{A_t Q_t} + \mathbb{E}_t \left[\frac{d(\bar{p} A_t Q_t)}{\bar{p} A_t Q_t} \frac{1}{dt} \right] = \rho + \underbrace{\frac{L_t}{\bar{p} A_t Q_t}}_{>0} + \mathbb{E}_t \left[\frac{d(\bar{p} A_t Q_t)}{\bar{p} A_t Q_t} \frac{1}{dt} \right] .$$

Assuming capitalists finance this transfer L_t by paying a portion φ of their wealth a_t, the dividend yield increases to $\rho + \varphi$ from a baseline yield (before transfers) of ρ. This adjustment raises the effective natural rate of interest from $\underline{r}$ to $\underline{r} + \varphi$, resulting in an increase in asset prices $\hat{Q}_t$ and a narrower output gap during a ZLB episode. Proposition 6 summarizes this result.

Proposition 6 (Fiscal Redistribution) *In the ZLB environment presented in Section 3, and under a redistribution scheme where a $\varphi \geq 0$ portion of capitalists' wealth is transferred to hand-to-mouth workers, the dynamic IS equation for $\hat{Q}_t$ becomes:*

$$d\hat{Q}_t = -(\underbrace{\underline{r}}_{<0} + \varphi)\, dt \,, \tag{25}$$

for $\underline{r} + \varphi < 0$. After time T, the central bank perfectly stabilizes the economy and eliminates the volatility in asset prices, $\sigma_t^q = 0$, for all $t \geq T$. When $\underline{r} + \varphi > 0$, fiscal transfers lift the economy out of the ZLB and immediate stabilization is possible by adhering to the policy rule outlined in equation (11), with $\underline{r} + \varphi$ as the effective natural rate.

Proof. See Appendix II.

■

From the capitalists' perspective, this policy effectively reduces their expected wealth growth by φ, taking the expected stock market return i_t^m as given. At the ZLB, i_t^m does not react to fiscal transfers due to the the binding constraint on the policy rate i_t.[26] As a result, the equilibrium growth rates of capitalists' wealth and the stock price index fall by φ, due to a less significant initial decline in asset prices $\hat{Q}_0$ at the start of the ZLB episode. Therefore, fiscal transfers to workers with a high marginal propensity to consume not only enhance aggregate demand but also create additional wealth effects which manifest through increases in dividend yields and asset prices, $\hat{Q}_t$.

[26] Note from the capitalists' optimization that risk-premium rp_t is given by $\bar{\sigma}^2$ during the ZLB, and $i_t^m = i_t + \text{rp}_t$.

6 Conclusion

This paper begins by exploring the likelihood of increased financial volatility at the ZLB, and finds that a credible commitment to economic stabilization in the future prevents excess financial volatility from developing. Next, we examine the effects of traditional forward guidance, defined by the monetary authority's promise to maintain a zero policy rate for an extended period of time. We show that this commitment fosters expectations of higher future asset prices and aggregate demand, thereby increasing the market valuation of households' financial wealth and, consequently, their aggregate consumption at the ZLB.

Our findings also suggest that a central bank might not always find it optimal to commit to perfectly stabilizing the business cycle in the future. By refusing to do so, the central bank opens up the possibility for alternative equilibrium paths with lower financial volatility at the ZLB and higher expected welfare. While this strategy is preferable from a welfare perspective, it involves trade-offs. Specifically, a lack of a commitment to future stabilization allows the central bank to reduce financial volatility at the ZLB, but at the expense of potentially large and costly output gap deviations in the future.

Finally, our analysis investigates the efficacy of alternative fiscal policies at the ZLB, such as subsidies for risky asset investments and fiscal redistribution among households. We show that both policies have the potential to augment the households' risk bearing capacity, resulting in a higher valuation of their financial wealth and consequently, an increase in aggregate consumption demand.

This paper aims to provide valuable insights for academics and policymakers interested in the interplay between financial uncertainty and unconventional policies at the ZLB, notably forward guidance. We leave to future research the study of central banks' communications policy under alternative scenarios, such as private information about the state of the economy held by the central bank.

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I Parameter Calibration

	Parameter Description	Value	Source
φ	Relative Risk Aversion	0.2	Within the admissible calibration ranges specified by Gandelman and Hernández-Murillo (2014).
χ_0	Inverse Frisch labor supply elasticity	0.25	See King and Rebelo (1999).
ρ	Subjective time discount factor	0.020	Target 2.8% natural rate.
g	TFP growth rate	0.0083	Annual growth rate of 3.3%, which corresponds to the US TFP growth rate from 2009 to 2020, as detailed in Table 8 of Comin et al. (2023).
$\underline{\sigma}$	TFP volatility, low volatility regime	0.009	See Dordal i Carreras et al. (2016).
$\bar{\sigma}$	TFP volatility, high volatility regime	0.209	Target -1.5% natural rate (ZLB recession).
T	ZLB duration (quarters)	20	A five-year ZLB duration, consistent with periods such as the Global Financial Crisis and the Great Recession. See Dordal i Carreras et al. (2016).
ν	Stabilization probability parameter	1	Target average duration $1/\nu$ of one quarter before returning to stabilization.
α	$1-$ Labor income share	0.4	See Alvarez-Cuadrado et al. (2018).
ϵ	Elasticity of substitution intermediate goods	7	Target steady-state mark-up of 16.7%. See Galí (2015).

Table I.1: Parameter calibration used in Section 4.

II Proofs and Derivations

Proof of Proposition 2. In the context outlined in Section 4.2, the central bank solves the following problem:[1]

$$\min_{\sigma_1^{q,L},\sigma_2^{q,L},\hat{T}^{\mathrm{HOFG}}} \mathbb{E}_0 \int_0^\infty e^{-\rho t}\hat{Q}_t^2 dt \ , \quad \text{s.t. } d\hat{Q}_t = \begin{cases} -\underbrace{r_1^T(\sigma_1^{q,L})}_{<0} dt + \sigma_1^{q,L} dZ_t \ , & \text{for } t < T \ , \\ -\underbrace{r_2^T(\sigma_2^{q,L})}_{>0} dt + \sigma_2^{q,L} dZ_t \ , & \text{for } T \le t < \hat{T}^{\mathrm{HOFG}} \ , \\ 0 \ , & \text{for } t \ge \hat{T}^{\mathrm{HOFG}} \ , \end{cases}$$

$$\text{with } \hat{Q}_0 = r_1^T(\sigma_1^{q,L})T + r_2^T(\sigma_2^{q,L})(\hat{T}^{\mathrm{HOFG}} - T) \ , \tag{II.1}$$

where

$$r_1^T(\sigma_1^{q,L}) \equiv \rho + g - \frac{\bar{\sigma}^2}{2} - \frac{(\bar{\sigma} + \sigma_1^{q,L})^2}{2} < 0 \ , \ r_2^T(\sigma_2^{q,L}) \equiv \rho + g - \frac{\underline{\sigma}^2}{2} - \frac{(\underline{\sigma} + \sigma_2^{q,L})^2}{2} > 0 \ .$$

After $\hat{T}^{\mathrm{HOFG}}$, there are no additional fluctuation in $\hat{Q}_t$. Defining r_s^T as $r_1^T(\sigma_1^{q,L})$ for $s < T$ and as $r_2^T(\sigma_2^{q,L})$ for $T \le s \le \hat{T}^{\mathrm{HOFG}}$, the process of $\hat{Q}_t$ can be articulated as follows:

$$\hat{Q}_t = \begin{cases} \underbrace{\int_t^{\hat{T}^{\mathrm{HOFG}}} r_s^T ds}_{\equiv \hat{Q}_{\mathrm{d}}(t;\hat{T}^{\mathrm{HOFG}})} + \sigma_1^{q,L} \underbrace{Z_t}_{\sim N(0,t)} \ , & \text{for } t \le T \ , \\ \underbrace{\int_t^{\hat{T}^{\mathrm{HOFG}}} r^T(s) ds}_{\equiv \hat{Q}_{\mathrm{d}}(t;\hat{T}^{\mathrm{HOFG}})} + \sigma_1^{q,L} Z_T + \sigma_2^{q,L} \underbrace{W_{t-T}}_{\sim N(0,t-T)} \ , & \text{for } T < t \le \hat{T}^{\mathrm{HOFG}} \ , \\ \sigma_1^{q,L} Z_T + \sigma_2^{q,L} \underbrace{W_{\hat{T}-T}}_{\sim N(0,\hat{T}-T)} = \hat{Q}_{\hat{T}^{\mathrm{HOFG}}} , & \text{for } \hat{T}^{\mathrm{HOFG}} < t \ . \end{cases} \tag{II.2}$$

where it is assumed that after $\hat{T}^{\mathrm{HOFG}}$, central banks maintain $\sigma_t^q = \sigma_t^{q,n} = 0$. In this equation, Z_t, W_{t-T}, and $U_{\hat{T}-T}$ are independent Brownian motions. If we square each term in equation (II.2) and apply the expectation operator with respect to the information

[1] For this proof, it is implicitly assumed that $r_1^T(\sigma_1^{q,L}) < 0$ and $r_2^T(\sigma_2^{q,L}) > 0$ hold for the optimal values of $\sigma_1^{q,L}$ and $\sigma_2^{q,L}$, ensuring that the ZLB remains effective up to time T.

available at $t = 0$, we obtain:

$$\mathbb{E}_0\, \hat{Q}_t^2 = \begin{cases} \hat{Q}_{\text{d}}(t; \hat{T}^{\text{HOFG}})^2 + \left(\sigma_1^{q,L}\right)^2 t\,, & \text{for } \ t \leq T\ , \\ \hat{Q}_{\text{d}}(t; \hat{T}^{\text{HOFG}})^2 + \left(\sigma_1^{q,L}\right)^2 T + \left(\sigma_2^{q,L}\right)^2 (t - T)\,, & \text{for } \ T < t \leq \hat{T}^{\text{HOFG}}\ , \\ \left(\sigma_1^{q,L}\right)^2 T + \left(\sigma_2^{q,L}\right)^2 (\hat{T}^{\text{HOFG}} - T)\,, & \text{for } \ \hat{T}^{\text{HOFG}} < t\ . \end{cases} \tag{II.3}$$

If we substitute equation (II.3) into the central bank's loss function (14), the central bank's commitment problem can be expressed as follows:

$$\begin{aligned} &\min_{\hat{T}^{\text{HOFG}}, \sigma_1^{q,L}, \sigma_2^{q,L}} \ \mathbb{E}_0 \int_0^\infty e^{-\rho t} \hat{Q}_t^2\, dt \\ &= \min_{\hat{T}^{\text{HOFG}}, \sigma_1^{q,L}, \sigma_2^{q,L}} \ \int_0^{\hat{T}} e^{-\rho t} \hat{Q}_{\text{d}}(t; \hat{T}^{\text{HOFG}})^2 dt + \left(\sigma_1^{q,L}\right)^2 \underbrace{\int_0^T t e^{-\rho t} dt}_{= \frac{1}{\rho^2} - \frac{1}{\rho^2} e^{-\rho T} - \cancel{\frac{T}{\rho} e^{-\rho T}}} + \left(\sigma_1^{q,L}\right)^2 T \underbrace{\int_T^\infty e^{-\rho t} dt}_{= \cancel{\frac{1}{\rho} e^{-\rho T}}} \\ &\quad + \left(\sigma_2^{q,L}\right)^2 \underbrace{\int_T^{\hat{T}^{\text{HOFG}}} e^{-\rho t} (t - T) dt}_{= \cancel{-\frac{1}{\rho} (\hat{T}^{\text{HOFG}} - T) e^{-\rho \hat{T}^{\text{HOFG}}}} + \frac{e^{-\rho T} - e^{-\rho \hat{T}^{\text{HOFG}}}}{\rho^2}} + \left(\sigma_2^{q,L}\right)^2 (\hat{T}^{\text{HOFG}} - T) \underbrace{\int_{\hat{T}^{\text{HOFG}}}^\infty e^{-\rho t} dt}_{= \cancel{\frac{1}{\rho} e^{-\rho \hat{T}^{\text{HOFG}}}}} \\ &= \min_{\hat{T}, \sigma_1^{q,L}, \sigma_2^{q,L}} \ \underbrace{\int_0^{\hat{T}^{\text{HOFG}}} e^{-\rho t} \hat{Q}_{\text{d}}(t; \hat{T}^{\text{HOFG}})^2 dt}_{\text{Deterministic fluctuations}} + \underbrace{\left(\sigma_1^{q,L}\right)^2 \frac{1}{\rho^2} (1 - e^{-\rho T}) + \left(\sigma_2^{q,L}\right)^2 \left(\frac{e^{-\rho T} - e^{-\rho \hat{T}^{\text{HOFG}}}}{\rho^2} \right)}_{\text{Stochastic fluctuations}}\,. \end{aligned} \tag{II.4}$$

The central bank now has control over $\sigma_1^{q,L}$, $\sigma_2^{q,L}$, and $\hat{T}^{\text{HOFG}}$, in addition to its conventional monetary policy tool $\{i_t\}$. Initially, we derive the first-order condition for $\hat{T}^{\text{HOFG}}$, which is as follows:

$$2 \cdot \underbrace{r_2^T(\sigma_2^{q,L})}_{>0} \int_0^{\hat{T}^{\text{HOFG}}} e^{-\rho t} \hat{Q}_{\text{d}}(t; \hat{T}^{\text{HOFG}}) dt + \left(\sigma_2^{q,L}\right)^2 \frac{1}{\rho} e^{-\rho \hat{T}^{\text{HOFG}}} = 0\,, \tag{II.5}$$

from which we obtain

$$\int_0^\infty e^{-\rho t} \hat{Q}_{\text{d}}(t; \hat{T}^{\text{HOFG}}) dt = \int_0^{\hat{T}^{\text{HOFG}}} e^{-\rho t} \hat{Q}_{\text{d}}(t; \hat{T}^{\text{HOFG}} \| \sigma_1^{q,L} < 0, \sigma_2^{q,L} < 0) dt < 0\,. \tag{II.6}$$

The first-order condition for $\hat{T}^{\text{HOFG}}$ indicates that, at the optimum, the central bank reduces the value of $\hat{T}^{\text{HOFG}}$ compared to $\hat{T}^{\text{TFG}}$ (traditional forward guidance), as discussed in Section 4.1. This is because when the central bank utilizes traditional forward guidance and achieves perfect stabilization for $t \geq \hat{T}^{\text{TFG}}$, the expression above becomes

$$\int_0^{\hat{T}^{\text{TFG}}} e^{-\rho t}\hat{Q}_{\text{d}}(t;\hat{T}\|\sigma_1^{q,L}=\sigma_1^{q,n}=0,\sigma_2^{q,L}=\sigma_2^{q,n}=0)dt = 0\ , \tag{II.7}$$

which is derived by plugging $\sigma_1^{q,L}=0$ and $\sigma_2^{q,L}=0$ into equation (II.5).

Given that at the optimum, $\sigma_1^{q,L}<0$ and $\sigma_2^{q,L}<0$ (which we will demonstrate),

$$\hat{Q}_{\text{d}}(t;\hat{T}^{\text{HOFG}}\|\sigma_1^{q,L}=0,\sigma_2^{q,L}=0) < \hat{Q}_{\text{d}}(t;\hat{T}^{\text{HOFG}}\|\sigma_1^{q,L}<0,\sigma_2^{q,L}<0)\ .$$

Therefore, we deduce from equation (II.1) that at the optimum, $\hat{T}^{\text{HOFG}} < \hat{T}^{\text{TFG}}$, as evidenced by comparing (II.7) with (II.6).

To characterize the optimal values of $\sigma_1^{q,L}$ and $\sigma_2^{q,L}$, a **variational argument** is required. This is because $\sigma_1^{q,L}$ and $\sigma_2^{q,L}$ influence the levels of $r_1^T(\sigma_1^{q,L})$, $r_2^T(\sigma_2^{q,L})$, and $\hat{Q}_{\text{d}}(t;\hat{T}^{\text{HOFG}})$. Specifically, we can derive:

$$\frac{\partial r_1^T(\sigma_1^{q,L})}{\partial \sigma_1^{q,L}} = -\left(\bar{\sigma}+\sigma_1^{q,L}\right)<0,\ \ \frac{\partial r_2^T(\sigma_2^{q,L})}{\partial \sigma_2^{q,L}} = -\left(\underline{\sigma}+\sigma_2^{q,L}\right)<0\ .$$

Determining $\sigma_1^{q,L}$ An increase in $\sigma_1^{q,L}$ leads to a decrease in $r_1^T(\sigma_1^{q,L})$, which alters the trajectory of $\hat{Q}_{\text{d}}(t;\hat{T}^{\text{HOFG}})$. This change is illustrated in Figure II.1, as depicted by the transition from the thick blue line to the dashed red line.

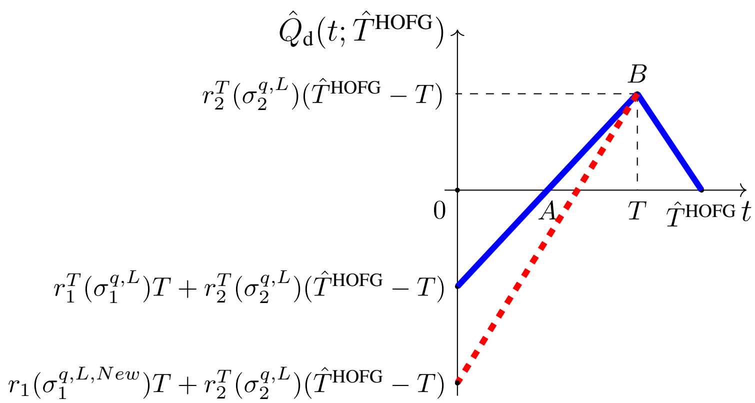

Figure II.1: Variation along $\sigma_1^{q,L}$. Increase to $\sigma_1^{q,L,New}>\sigma_1^{q,L}$.

Differentiating $\hat{Q}_{\mathrm{d}}(t;\hat{T}^{\mathrm{HOFG}}) = \int_t^{\hat{T}^{\mathrm{HOFG}}} r_s^T ds$ with respect to $\sigma_1^{q,L}$, we obtain:

$$\frac{\partial \hat{Q}_{\mathrm{d}}(t;\hat{T}^{\mathrm{HOFG}})}{\partial \sigma_1^{q,L}} = \int_t^T -\left(\bar{\sigma} + \sigma_1^{q,L}\right) ds = -\left(\bar{\sigma} + \sigma_1^{q,L}\right)(T-t), \ \ \forall t \leq T \ .$$

To find optimal $\sigma_1^{q,L}$, we differentiate the objective function in (II.4) by $\sigma_1^{q,L}$ and obtain the following condition:

$$\left(\bar{\sigma} + \sigma_1^{q,L}\right) \int_0^T e^{-\rho t} \hat{Q}_{\mathrm{d}}(t;\hat{T}^{\mathrm{HOFG}})(T-t)dt = \left(\sigma_1^{q,L}\right) \frac{1-e^{-\rho T}}{\rho^2} \ ,$$

from which we can prove that $\sigma_1^{q,L} < 0$ must be satisfied at the optimum, given that

$$\int_0^T e^{-\rho t} \hat{Q}_{\mathrm{d}}(t;\hat{T}^{\mathrm{HOFG}})(T-t)dt = \underbrace{\int_0^t e^{-\rho s} \hat{Q}_{\mathrm{d}}(s;\hat{T}^{\mathrm{HOFG}})ds(T-t)\Big|_0^T}_{=0} + \int_0^T \underbrace{\int_0^t e^{-\rho s} \hat{Q}_{\mathrm{d}}(s;\hat{T}^{\mathrm{HOFG}})ds\, dt}_{<0} < 0 \ ,$$

where $\int_0^t e^{-\rho s} \hat{Q}_{\mathrm{d}}(s;\hat{T}^{\mathrm{HOFG}})ds < 0$ for $t \leq T$, as derived in equation (II.6).

Determining $\sigma_2^{q,L}$ An increase in $\sigma_2^{q,L}$ leads to a decrease in $r_2^T(\sigma_2^{q,L})$, which alters the shape of $\hat{Q}_{\mathrm{d}}(t;\hat{T}^{\mathrm{HOFG}})$. This effect is illustrated in Figure II.2 by the transition from the thick blue line to the dashed red line. To further analyze this, we differentiate $\hat{Q}_{\mathrm{d}}(t;\hat{T}^{\mathrm{HOFG}})$ with respect to $\sigma_2^{q,L}$ and obtain:

$$\frac{\partial \hat{Q}_{\mathrm{d}}(t;\hat{T}^{\mathrm{HOFG}})}{\partial \sigma_2^{q,L}} = \begin{cases} \int_T^{\hat{T}^{\mathrm{HOFG}}} -\left(\underline{\sigma} + \sigma_2^{q,L}\right) ds = -\left(\underline{\sigma} + \sigma_2^{q,L}\right)(\hat{T}^{\mathrm{HOFG}} - T), & t < T \ , \\ \int_t^{\hat{T}^{\mathrm{HOFG}}} -\left(\underline{\sigma} + \sigma_2^{q,L}\right) ds = -\left(\underline{\sigma} + \sigma_2^{q,L}\right)(\hat{T}^{\mathrm{HOFG}} - t), & T \leq t \leq \hat{T}^{\mathrm{HOFG}} \ . \end{cases}$$

To find the optimal $\sigma_2^{q,L}$, we differentiate the objective function in (II.4) by $\sigma_2^{q,L}$ and obtain

$$\left(\underline{\sigma} + \sigma_2^{q,L}\right)\left(\int_0^T e^{-\rho t} \hat{Q}_{\mathrm{d}}(t;\hat{T}^{\mathrm{HOFG}})(\hat{T}^{\mathrm{HOFG}} - T)dt + \int_T^{\hat{T}^{\mathrm{HOFG}}} e^{-\rho t} \underbrace{\hat{Q}_{\mathrm{d}}(t;\hat{T}^{\mathrm{HOFG}})}_{>0}(\hat{T}^{\mathrm{HOFG}} - t)dt\right)$$
$$= (\sigma_2^{q,L}) \frac{e^{-\rho T} - e^{-\rho \hat{T}}}{\rho^2} \ ,$$

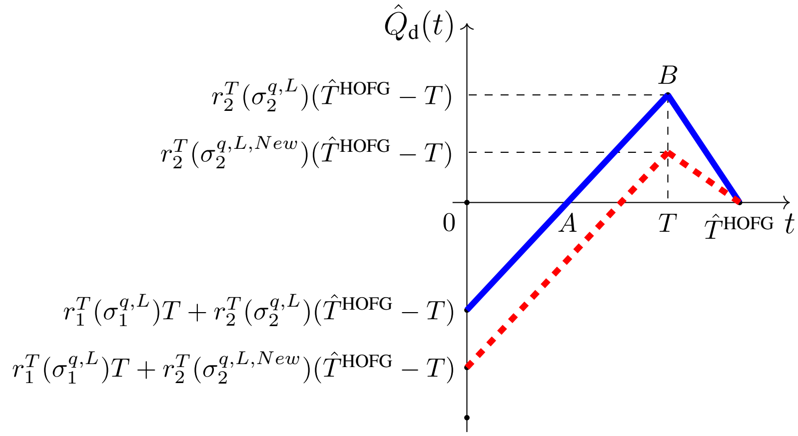

Figure II.2: Variation along $\sigma_2^{q,L}$. Increase to $\sigma_2^{q,L,New} > \sigma_2^{q,L}$.

from which we can demonstrate that at the optimum, $\sigma_2^{q,L} < 0$ must be satisfied, given that

$$
\begin{aligned}
&\int_0^T e^{-\rho t}\hat{Q}_{\text{d}}(t;\hat{T}^{\text{HOFG}})(\hat{T}^{\text{HOFG}} - T)dt + \int_T^{\hat{T}^{\text{HOFG}}} e^{-\rho t}\underbrace{\hat{Q}_{\text{d}}(t;\hat{T}^{\text{HOFG}})}_{>0}(\hat{T}^{\text{HOFG}} - t)dt \\
&< \int_0^T e^{-\rho t}\hat{Q}_{\text{d}}(t;\hat{T}^{\text{HOFG}})(\hat{T}^{\text{HOFG}} - T)dt + \int_T^{\hat{T}^{\text{HOFG}}} e^{-\rho t}\underbrace{\hat{Q}_{\text{d}}(t;\hat{T}^{\text{HOFG}})}_{>0}(\hat{T}^{\text{HOFG}} - T)dt \\
&= (\hat{T}^{\text{HOFG}} - T)\underbrace{\int_0^{\hat{T}^{\text{HOFG}}} e^{-\rho t}\hat{Q}_{\text{d}}(t;\hat{T}^{\text{HOFG}})dt}_{<0} < 0\,,
\end{aligned}
$$

where the final inequality is derived from equation (II.6). Hence, during periods of high TFP volatility (i.e., $t < T$) and low TFP volatility with forward guidance (i.e., $T \leq t \leq \hat{T}^{\text{HOFG}}$), a central bank aims to target financial volatility levels below those in a flexible price economy: $\sigma_1^{q,L} < \sigma_1^{q,n} = 0$ and $\sigma_2^{q,L} < \sigma_2^{q,n} = 0$. Such intervention reduces the required risk premium and raises the asset price level $\hat{Q}_t$, thereby increasing output.

■

First-Order Conditions for $\sigma_1^{q,L}$, $\sigma_2^{q,L}$, and $\hat{T}^{\text{HOFG}}$ The deterministic component of the capitalists' asset gap process $\hat{Q}_t$, denoted as $\hat{Q}_{\text{d}}(t;\hat{T}^{\text{HOFG}})$, is defined as follows (with

$r_1^T(\sigma_1^{q,L})$ and $r_2^T(\sigma_2^{q,L})$ specified in equation (16)):

$$\hat{Q}_{\mathrm{d}}(t;\hat{T}^{\mathrm{HOFG}}) = \int_t^{\hat{T}^{\mathrm{HOFG}}} r_s^T ds = \begin{cases} \underbrace{r_1^T(\sigma_1^{q,L})}_{<0}(T-t) + \underbrace{r_2^T(\sigma_2^{q,L})}_{>0}(\hat{T}^{\mathrm{HOFG}} - T), & \text{for } \forall t \leq T \ , \\ r_2^T(\sigma_2^{q,L})(\hat{T}^{\mathrm{HOFG}} - t), & \text{for } T \leq \forall t < \hat{T}^{\mathrm{HOFG}} \ , \end{cases}$$

from which we derive the following:

$$\begin{aligned} \int_0^{\hat{T}^{\mathrm{HOFG}}} e^{-\rho t}\hat{Q}_{\mathrm{d}}(t;\hat{T}^{\mathrm{HOFG}})dt = &\int_0^T e^{-\rho t}\left[r_1^T(\sigma_1^{q,L})(T-t) + r_2^T(\sigma_2^{q,L})(\hat{T}^{\mathrm{HOFG}} - T)\right] dt \\ &+ \int_T^{\hat{T}^{\mathrm{HOFG}}} e^{-\rho t} r_2^T(\sigma_2^{q,L})(\hat{T}^{\mathrm{HOFG}} - t)dt \ . \end{aligned} \tag{II.8}$$

The first condition for $\hat{T}^{\mathrm{HOFG}}$ can be written as

$$2 \cdot r_2^T(\sigma_2^{q,L}) \int_0^{\hat{T}^{\mathrm{HOFG}}} e^{-\rho t}\hat{Q}_{\mathrm{d}}(t;\hat{T}^{\mathrm{HOFG}})dt + \left(\sigma_2^{q,L}\right)^2 \frac{e^{-\rho \hat{T}^{\mathrm{HOFG}}}}{\rho} = 0 \ , \tag{II.9}$$

where

$$\begin{aligned} \int_0^{\hat{T}^{\mathrm{HOFG}}} e^{-\rho t}\hat{Q}_{\mathrm{d}}(t;\hat{T}^{\mathrm{HOFG}})dt = &r_1^T(\sigma_1^{q,L})\left[\frac{e^{-\rho T}}{\rho^2} + \frac{T}{\rho} - \frac{1}{\rho^2}\right] + r_2^T(\sigma_2^{q,L})(\hat{T}^{\mathrm{HOFG}} - T)\frac{1-e^{-\rho T}}{\rho} \\ &+ r_2^T(\sigma_2^{q,L})\left[\frac{e^{-\rho \hat{T}^{\mathrm{HOFG}}}}{\rho^2} + \frac{\hat{T}^{\mathrm{HOFG}} - T}{\rho}e^{-\rho T} - \frac{1}{\rho^2}e^{-\rho T}\right] , \end{aligned}$$

follows from equation (II.8). Combined with equation (II.9), the first-order condition for $\hat{T}^{\mathrm{HOFG}}$ is expressed as follows:

$$\begin{aligned} 2 \cdot r_2^T(\sigma_2^{q,L}) \Bigg[&r_1^T(\sigma_1^{q,L})\left[\frac{e^{-\rho T}}{\rho^2} + \frac{T}{\rho} - \frac{1}{\rho^2}\right] + r_2^T(\sigma_2^{q,L})(\hat{T}^{\mathrm{HOFG}} - T)\frac{1-e^{-\rho T}}{\rho} \\ &+ r_2^T(\sigma_2^{q,L})\left[\frac{e^{-\rho \hat{T}^{\mathrm{HOFG}}}}{\rho^2} + \frac{\hat{T}^{\mathrm{HOFG}} - T}{\rho}e^{-\rho T} - \frac{1}{\rho^2}e^{-\rho T}\right]\Bigg] + \left(\sigma_2^{q,L}\right)^2 \frac{e^{-\rho \hat{T}^{\mathrm{HOFG}}}}{\rho} = 0 \ . \end{aligned}$$

The first-order condition for $\sigma_1^{q,L}$ is expressed as

$$\left(\bar{\sigma} + \sigma_1^{q,L}\right) \int_0^T e^{-\rho t}\hat{Q}_{\mathrm{d}}(t;\hat{T}^{\mathrm{HOFG}})(T-t)dt = \left(\sigma_1^{q,L}\right)\frac{1-e^{-\rho T}}{\rho^2} \ , \tag{II.10}$$

where

$$\int_0^T e^{-\rho t}\hat{Q}_{\rm d}(t;\hat{T}^{\rm HOFG})(T-t)dt = r_1^T(\sigma_1^{q,L})\left[-\frac{2}{\rho^3}e^{-\rho T}+\frac{T^2}{\rho}-\frac{2T}{\rho^2}+\frac{2}{\rho^3}\right] + r_2^T(\sigma_2^{q,L})(\hat{T}^{\rm HOFG}-T)\left[\frac{e^{-\rho T}}{\rho^2}+\frac{T}{\rho}-\frac{1}{\rho^2}\right]. \tag{II.11}$$

Substituting equation (II.11) into equation (II.10), we arrive at:

$$(\bar{\sigma}+\sigma_1^{q,L})\left[r_1^T(\sigma_1^{q,L})\left[-\frac{2}{\rho^3}e^{-\rho T}+\frac{T^2}{\rho}-\frac{2T}{\rho^2}+\frac{2}{\rho^3}\right]+r_2^T(\sigma_2^{q,L})(\hat{T}^{\rm HOFG}-T)\left[\frac{e^{-\rho T}}{\rho^2}+\frac{T}{\rho}-\frac{1}{\rho^2}\right]\right] = (\sigma_1^{q,L})\frac{1-e^{-\rho T}}{\rho^2},$$

as the first-order condition for $\sigma_1^{q,L}$. Finally, the first-order condition for $\sigma_2^{q,L}$ is as follows:

$$\left(\underline{\sigma}+\sigma_2^{q,L}\right)\left((\hat{T}^{\rm HOFG}-T)\int_0^T e^{-\rho t}\hat{Q}_{\rm d}(t;\hat{T}^{\rm HOFG})dt+\int_T^{\hat{T}^{\rm HOFG}} e^{-\rho t}\hat{Q}_{\rm d}(t;\hat{T}^{\rm HOFG})(\hat{T}^{\rm HOFG}-t)dt\right) = (\sigma_2^{q,L})\frac{e^{-\rho T}-e^{-\rho\hat{T}^{\rm HOFG}}}{\rho^2},$$

Therefore, the first-order condition for $\sigma_2^{q,L}$ is expressed as:[2]

$$\left(\underline{\sigma}+\sigma_2^{q,L}\right)\left[\left[r_1^T(\sigma_1^{q,L})\left[\frac{e^{-\rho T}}{\rho^2}+\frac{T}{\rho}-\frac{1}{\rho^2}\right]+r_2^T(\sigma_2^{q,L})(\hat{T}^{\rm HOFG}-T)\frac{1-e^{-\rho T}}{\rho}\right](\hat{T}^{\rm HOFG}-T) + r_2^T(\sigma_2^{q,L})\left[-\frac{2}{\rho^3}e^{-\rho\hat{T}^{\rm HOFG}}+\frac{(\hat{T}^{\rm HOFG}-T)^2}{\rho}e^{-\rho T}-\frac{2(\hat{T}^{\rm HOFG}-T)}{\rho^2}e^{-\rho T}+\frac{2}{\rho^3}e^{-\rho T}\right]\right] = \left(\sigma_2^{q,L}\right)\frac{e^{-\rho T}-e^{-\rho\hat{T}^{\rm HOFG}}}{\rho^2}.$$

[2] We use the following properties of $\hat{Q}_d\left(t;\hat{T}^{\rm HOFG}\right)$:

$$\int_0^T e^{-\rho t}\hat{Q}_{\rm d}(t;\hat{T}^{\rm HOFG})dt = r_1^T(\sigma_1^{q,L})\left[\frac{e^{-\rho T}}{\rho^2}+\frac{T}{\rho}-\frac{1}{\rho^2}\right]+r_2^T(\sigma_2^{q,L})(\hat{T}^{\rm HOFG}-T)\frac{1-e^{-\rho T}}{\rho},$$

and

$$\int_T^{\hat{T}^{\rm HOFG}} e^{-\rho t}\hat{Q}_{\rm d}(t;\hat{T}^{\rm HOFG})(\hat{T}^{\rm HOFG}-t)dt = r_2^T(\sigma_2^{q,L})\left[-\frac{2e^{-\rho\hat{T}^{\rm HOFG}}}{\rho^3}+\frac{(\hat{T}^{\rm HOFG}-T)^2}{\rho}e^{-\rho T}-\frac{2(\hat{T}^{\rm HOFG}-T)}{\rho^2}e^{-\rho T}+\frac{2e^{-\rho T}}{\rho^3}\right].$$

Proof of Proposition 4. We begin by solving the capitalist's problem presented in equation (20), considering a subsidy rate τ on stock market investments for $t \leq T$:

$$\begin{aligned}
&\max_{C_t,\theta_t} \mathbb{E}_0 \int_0^{\infty} e^{-\rho t} \log C_t dt \\
&\text{s.t. } da_t = (a_t(i_t + \theta_t((1+\tau)\, i_t^m - i_t)) - \bar{p}C_t - L_t)dt + \theta_t a_t \left(\bar{\sigma} + \sigma_t^q\right) dZ_t \ .
\end{aligned} \tag{II.12}$$

Since the subsidy τ is financed through a lump-sum tax on capitalists, the dividend process in equation (4) and the stock market valuation equation (5) remain unchanged. As a result, $\bar{p}C_t = \rho a_t$ and $C_t = \rho A_t Q_t$. Equilibrium taxes L_t equal to $\tau i_t^m a_t$, and the budget constraint in equation (II.12) becomes

$$\begin{aligned}
\frac{dC_t}{C_t} = \frac{da_t}{a_t} &= ((1+\cancel{\tau})i_t^m - \rho - \cancel{\tau i_t^m})dt + (\bar{\sigma} + \sigma_t^q)dZ_t \\
&= (i_t^m - \rho)dt + (\bar{\sigma} + \sigma_t^q)dZ_t \ ,
\end{aligned} \tag{II.13}$$

where we used equilibrium condition $\theta_t = 1$. Since $\xi_t^N = e^{-\rho t}\frac{1}{\bar{p}C_t}$, we obtain:

$$\begin{aligned}
\frac{d\xi_t^N}{\xi_t^N}(i_t^m, \sigma_t^q) &= -\rho dt - \frac{dC_t}{C_t} + \left(\frac{dC_t}{C_t}\right)^2 \\
&= -\cancel{\rho} dt - [(i_t^m - \cancel{\rho})dt + (\bar{\sigma} + \sigma_t^q)dZ_t] + (\bar{\sigma} + \sigma_t^q)^2 dt \\
&= -\left[i_t^m - (\bar{\sigma} + \sigma_t^q)^2\right] dt - (\bar{\sigma} + \sigma_t^q)dZ_t \ .
\end{aligned} \tag{II.14}$$

The subsidy τ on the expected return i_t^m alters the original Euler equation $\mathbb{E}_t \frac{d\xi_t^N}{\xi_t^N} = -i_t dt$. Consequently, the revised expression with a subsidy τ must be :

$$\mathbb{E}_t\left[\frac{d\xi_t^N}{\xi_t^N}((1+\tau)i_t^m, \sigma_t^q)\right] = -\left[(1+\tau)i_t^m - (\bar{\sigma} + \sigma_t^q)^2\right] = -i_t dt \ ,$$

from which we obtain equation (21):

$$i_t^m = \frac{i_t + (\bar{\sigma} + \sigma_t^q)^2}{1+\tau} = \frac{\bar{\sigma}^2}{1+\tau} \ ,$$

where the final equality results from substituting $i_t = 0$ and $\sigma_t^q = 0$ into the equation. From equation (II.13),it follows that:

$$\frac{dC_t}{C_t} = (i_t^m - \rho)dt + \bar{\sigma}dZ_t = \left(\frac{\bar{\sigma}^2}{1+\tau} - \rho\right) dt + \bar{\sigma}dZ_t \ , \tag{II.15}$$

with which we obtain

$$d \ln C_t = \left(\frac{\bar{\sigma}^2}{1+\tau} - \rho - \frac{\bar{\sigma}^2}{2} \right) dt + \bar{\sigma} dZ_t \ .$$

Finally, by using equation (A.6) from Online Appendix A, we derive the natural counterpart to the above expression:

$$d \ln C_t^n = \Bigg(\underbrace{\bar{r}}_{<0} - \rho + \frac{\bar{\sigma}^2}{2} \Bigg) + \bar{\sigma} dZ_t \ . \tag{II.16}$$

Combining both expressions, we obtain the dynamic IS equation in (22).

■

Proof of Proposition 5. By equation (24), the condition that characterizes the equilibrium stock market return i_t^m is given by:

$$i_t^m = \frac{y_t - \overbrace{\frac{w_t}{\bar{p}} N_{W,t}}^{=C_{W,t} + \frac{L_t}{\bar{p}}}}{A_t Q_t} + \frac{d(\bar{p} A_t Q_t)}{\bar{p} A_t Q_t} \frac{1}{dt} = \underbrace{\rho - \tau i_t^m}_{\text{Dividend yield}} + \frac{d(\bar{p} A_t Q_t)}{\bar{p} A_t Q_t} \frac{1}{dt} \ ,$$

from which we obtain $(1+\tau) i_t^m = \rho + g + \mu_t^q$ using $\sigma_t^q = 0$. Since $(1+\tau) i_t^m = \bar{\sigma}^2$ by equation (21), we infer that μ_t^q remains constant in comparison to the scenario without subsidy, conditional on $i_t = 0$ and $\sigma_t^q = 0$. Therefore, the subsidy policy does not alter the $\{\hat{Q}_t\}$ process. To align this intuition with the mathematical representation, we begin by examining the process for C_t, which is different from that in equation (II.15), as capitalists are now exempt from paying taxes L_t:

$$\begin{aligned} \frac{dC_t}{C_t} &= ((1+\tau) i_t^m - \rho) dt + \bar{\sigma} dZ_t \\ &= (\bar{\sigma}^2 - \rho) dt + \bar{\sigma} dZ_t \ . \end{aligned}$$

Given that the previous expression remains unchanged in the presence of subsidy τ, it can be inferred that a policy subsidizing the expected return of the stock market and financed by a lump-sum tax on workers does not impact the $\{\hat{Q}_t\}$ process. Consequently, the dynamics of $\{\hat{Q}_t\}$ are identical to those in an economy without this policy.

■

Proof of Proposition 6. A fiscal transfer $L_t > 0$ from capitalists to hand-to-mouth workers increases the aggregate dividends in the financial market. This results in a reduced need for expected future capital gains, which translates into higher asset prices $\hat{Q}_t$ at the ZLB. The expected stock market return i_t^m under these circumstances is given by:

$$
\begin{aligned}
i_t^m &= \frac{A_t N_{W,t} - \overbrace{\frac{w_t}{\bar{p}}}^{=C_{W,t}-\frac{L_t}{\bar{p}}} N_{W,t}}{A_t Q_t} + \frac{d(\cancel{\bar{p}} A_t Q_t)}{\cancel{\bar{p}} A_t Q_t}\frac{1}{dt} = \rho + \underbrace{\frac{L_t}{\bar{p} A_t Q_t}}_{>0} + \frac{d(\cancel{\bar{p}} A_t Q_t)}{\cancel{\bar{p}} A_t Q_t}\frac{1}{dt} \\
&= \rho + \varphi + \frac{d(\cancel{\bar{p}} A_t Q_t)}{\cancel{\bar{p}} A_t Q_t}\frac{1}{dt} \ ,
\end{aligned}
$$

where the last equality follows from L_t being equal to $\varphi \bar{p} A_t Q_t$ in equilibrium.

To derive equation (25), we start from the capitalists' optimization problem:

$$
\begin{aligned}
&\max_{C_t, \theta_t} \mathbb{E}_0 \int_0^\infty e^{-\rho t} \log C_t dt \\
&\text{s.t. } da_t = (a_t(i_t + \theta_t(i_t^m - i_t)) - \bar{p} C_t - L_t)dt + \theta_t a_t(\bar{\sigma} + \sigma_t^q)dZ_t \ ,
\end{aligned}
$$

which features equilibrium conditions for C_t and θ_t identical to those described in equations (5) and (6), together with $\sigma_t^q = 0$. As a result, $C_t = \rho \bar{p} A_t Q_t$ and $i_t^m = i_t + (\bar{\sigma} + \sigma_t^q)^2$ follows. In an equilibrium where $\sigma_t^q = 0$ and i_t is constrained by the ZLB, the wealth process for capitalists is given by:

$$
\frac{dC_t}{C_t} = \frac{da_t}{a_t} = (i_t^m - \rho - \varphi)\, dt + \bar{\sigma}_t dZ_t = (\bar{\sigma}^2 - \varphi - \rho)dt + \bar{\sigma}_t dZ_t \ ,
$$

from which we derive

$$
d \ln C_t = \left(\frac{\bar{\sigma}^2}{2} - \varphi - \rho \right) dt + \bar{\sigma}_t dZ_t \ .
$$

Subtracting the process for C_t^n in equation (II.16) yields the dynamic IS equation in (25).

■

A Flexible Price Equilibrium

This section derives the flexible price equilibrium of the model, establishing it as the benchmark for economic and welfare analysis. We begin by revisiting the Fisherian identity, incorporating an inflation premium linked to wealth volatility into the relation. Lemma A.1 summarizes the modified identity.

Lemma A.1 (Inflation Premium) *The real interest rate of the economy is given by:*

$$r_t = i_t - \pi_t + \overbrace{\sigma_t^p \underbrace{\left(\sigma + \sigma_t^p + \sigma_t^q\right)}_{\textit{Wealth volatility}}}^{\textit{Inflation Premium}} \ . \tag{A.1}$$

Proof of Lemma A.1. The financial wealth of capitalists is equal to the value of the stock market index, $a_t = p_t A_t Q_t$, which follows from bonds being in zero net supply and capitalists being symmetric and identical in equilibrium. We start by stating capitalist's nominal state-price density ξ_t^N, which satisfies the following condition:

$$\frac{d\xi_t^N}{\xi_t^N} = -i_t dt - (\sigma + \sigma_t^q) dZ_t \ ,$$

and the real state price density ξ_t^r, which is given by

$$\xi_t^r = e^{-\rho t} \frac{1}{C_t} = p_t \xi_t^N \ . \tag{A.2}$$

Utilizing equations (2) and (3), and considering that $\theta_t = 1$ in equilibrium, the application of Ito's Lemma to equation (A.2) yields the following expression:

$$\frac{d\xi_t^r}{\xi_t^r} = \left(\underbrace{\pi_t - i_t - \sigma_t^p \left(\sigma + \sigma_t^q + \sigma_t^p\right)}_{=-r_t} \right) dt - (\sigma + \sigma_t^q) dZ_t \ ,$$

resulting in the modified Fisherian identity detailed in equation (A.1).

■

Definition A.1 *Let* $\chi^{-1} \equiv \dfrac{1-\varphi}{\chi_0+\varphi}$ *represent the effective labor supply elasticity of workers, conditional on their optimal consumption decision.*

Proposition A.1 summarizes the dynamics of the real wage, asset price, natural interest rate r_t^n, and the consumption process of capitalists within the flexible price equilibrium.

Proposition A.1 (Flexible Price Equilibrium) *In the flexible price equilibrium,*[1] *the following results are obtained:*

1. *The real wage is proportional to aggregate technology* A_t*, and given by*

$$\frac{w_t^n}{p_t} = \frac{(\epsilon-1)(1-\alpha)}{\epsilon} A_t \ .$$

2. *The equilibrium asset price* Q_t^n *is constant and given by*

$$Q_t^n = \frac{1}{\rho}\left(\frac{(\epsilon-1)(1-\alpha)}{\epsilon}\right)^{\frac{1}{\chi}}\left(1-\frac{(\epsilon-1)(1-\alpha)}{\epsilon}\right) \ , \quad \textit{and} \quad \mu_t^{q,n} = \sigma_t^{q,n} = 0 \ .$$

3. *The natural interest rate* r_t^n *is constant and defined as* $r_t^n \equiv r^n = \rho + g - \sigma^2$*. The consumption of capitalists evolves according to the following equation:*

$$\frac{dC_t^n}{C_t^n} = g dt + \sigma dZ_t = \underbrace{\left(r^n - \rho + \sigma^2\right)}_{\equiv \mu_t^{c,n}} dt + \underbrace{\sigma}_{\equiv \sigma_t^{c,n}} dZ_t \ .$$

Proof of Proposition A.1. Starting with the optimization problem of intermediate firms, the presence of an externality à la Baxter and King (1991) imposes extra steps on the aggregation process of individual decisions across firms. Utilizing the production function, the employed labor of firm i can be expressed as

$$n_t(i) = \left(\frac{y_t(i)}{A_t E_t}\right)^{\frac{1}{1-\alpha}} \ ,$$

[1] Variables in the flexible price (i.e., natural) equilibrium are denoted with the superscript n.

where we defined $E_t \equiv (N_{W,t})^\alpha$. At any given time t, each intermediate firm i determines the optimal price $p_t(i)$ to maximize its profits,

$$\max_{p_t(i)} p_t(i) \left(\frac{p_t(i)}{p_t}\right)^{-\epsilon} y_t - w_t \left(\frac{y_t}{A_t E_t}\right)^{\frac{1}{1-\alpha}} \left(\frac{p_t(i)}{p_t}\right)^{-\frac{\epsilon}{1-\alpha}} , \tag{A.3}$$

taking the aggregate demand of the economy y_t as given. In the flexible price equilibrium, all firms charge the same price, $p_t(i) = p_t$ for all i, and hire the same amount of labor, $n_t(i) = N_{w,t}$ for all i. From the first-order condition (A.3), we obtain the real wage as

$$\frac{w_t^n}{p_t} = \frac{\epsilon-1}{\epsilon}(1-\alpha) y_t^{\frac{-\alpha}{1-\alpha}} (A_t)^{\frac{1}{1-\alpha}} N_{W,t}{}^{\frac{\alpha}{1-\alpha}} = \frac{\epsilon-1}{\epsilon}(1-\alpha) y_t^{\frac{-\alpha}{1-\alpha}} (A_t)^{\frac{1}{1-\alpha}} \left(\frac{w_t^n}{p_t^n}\right)^{\frac{\alpha}{\chi(1-\alpha)}} A_t^{\frac{-\alpha}{\chi(1-\alpha)}} ,$$

which can be further simplified to the following expression:

$$\frac{w_t^n}{p_t} = \left(\frac{\epsilon-1}{\epsilon}(1-\alpha)\right)^{\frac{\chi(1-\alpha)}{\chi(1-\alpha)-\alpha}} y_t^{\frac{-\chi\alpha}{\chi(1-\alpha)-\alpha}} A_t^{\frac{\chi-\alpha}{\chi(1-\alpha)-\alpha}} .$$

Aggregate production in the flexible price equilibrium is linear, $y_t = A_t N_{W,t}$. We obtain:

$$y_t = A_t \left(\frac{\epsilon-1}{\epsilon}(1-\alpha)\right)^{\frac{(1-\alpha)}{\chi(1-\alpha)-\alpha}} y_t^{\frac{-\alpha}{\chi(1-\alpha)-\alpha}} A_t^{\frac{1-\frac{\alpha}{\chi}}{\chi(1-\alpha)-\alpha}} A_t^{-\frac{1}{\chi}} .$$

The previous expression allows us to write the natural level of output y_t^n and the natural real wage $\frac{w_t^n}{p_t}$ as

$$y_t^n = \left(\frac{\epsilon-1}{\epsilon}(1-\alpha)\right)^{\frac{1}{\chi}} A_t \text{ and } \frac{w_t^n}{p_t} = \frac{\epsilon-1}{\epsilon}(1-\alpha) A_t ,$$

from which we obtain

$$N_{W,t}^n = \left(\frac{\epsilon-1}{\epsilon}(1-\alpha)\right)^{\frac{1}{\chi}} \text{ and } C_{W,t}^n = \left(\frac{\epsilon-1}{\epsilon}(1-\alpha)\right)^{1+\frac{1}{\chi}} A_t . \tag{A.4}$$

In equilibrium, the combined consumption of capitalists and workers equates to the total final output, as detailed in equation (7). Following from equation (A.4), we obtain:

$$\rho A_t Q_t^n + \left(\frac{\epsilon-1}{\epsilon}(1-\alpha)\right)^{1+\frac{1}{\chi}} A_t = \left(\frac{\epsilon-1}{\epsilon}(1-\alpha)\right)^{\frac{1}{\chi}} A_t \ .$$

where we defined Q_t^n to be the natural stock price. Therefore, we obtain an expression for Q_t^n as

$$Q_t^n = \frac{1}{\rho}\left(\frac{\epsilon-1}{\epsilon}(1-\alpha)\right)^{\frac{1}{\chi}}\left(1-\frac{(\epsilon-1)(1-\alpha)}{\epsilon}\right) \ ,$$

and $C_t^n = \rho A_t Q_t^n$. Since Q_t^n is constant in equilibrium, its process in a flexible price economy exhibits neither drift nor volatility, which implies $\mu_t^{q,n} = \sigma_t^{q,n} = 0$. To determine the natural interest rate r_t^n, we start from the capital gain component outlined in equation (8). The application of Ito's lemma yields:

$$\mathbb{E}_t \frac{d\left(p_t A_t Q_t\right)}{p_t A_t Q_t}\frac{1}{dt} = \pi_t + \underbrace{\mu_t^q}_{=0} + g + \underbrace{\sigma_t^q}_{=0}\sigma_t^p + \sigma\left(\sigma_t^p + \underbrace{\sigma_t^q}_{=0}\right) \ .$$

Given a constant dividend yield equal to ρ, applying expectations to both sides of equation (8) and combining this expression with the equilibrium condition presented in equation (6) results in:

$$i_t^m = \rho + \pi_t + g + \sigma\sigma_t^p = i_t + \left(\sigma + \sigma_t^p\right)^2 \ .$$

Inserting the previous expression into the Fisherian identity in equation (A.1), we express the natural rate of interest r_t^n as

$$r_t^n = i_t - \pi_t + \sigma_t^p\left(\sigma + \underbrace{\sigma_t^{q,n}}_{=0} + \sigma_t^p\right) = \rho + g - \sigma^2 \ , \tag{A.5}$$

which is a function of structural parameters, including σ, thereby proving the final point of

Proposition A.1. As the consumption of capitalists C_t^n is directly proportional to the level of technology A_t, it follows that:

$$\frac{dC_t^n}{C_t^n} = g dt + \sigma dZ_t = \left(r_t^n - \rho + \sigma^2\right) dt + \sigma dZ_t \ , \tag{A.6}$$

where the last equality is derived using equation equation (A.5).

■

B Co-movements between gap variables

The following Lemma B.2 demonstrates that Assumption B.1 serves as a sufficient condition for the model to exhibit the empirical regularities of positive co-movements between asset prices and various business cycle variables, such as real wage and consumption (of capitalists and workers), as observed in data.[2]

Assumption B.1 (Labor Supply Elasticity) *The effective labor supply elasticity of workers satisfies:* $\chi^{-1} > \frac{\frac{(\epsilon-1)(1-\alpha)}{\epsilon}}{1-\frac{(\epsilon-1)(1-\alpha)}{\epsilon}}$.

Lemma B.2 (Positive comovement) *Under Assumption B.1, the consumption gaps of capitalists C_t and workers $C_{W,t}$, employment $N_{W,t}$, and real wage $\frac{w_t}{p_t}$ exhibit joint positive comovement. This relationship is approximated up to a first-order as follows:*

$$\hat{Q}_t = \hat{C}_t = \underbrace{\left(\chi^{-1} - \frac{\frac{(\epsilon-1)(1-\alpha)}{\epsilon}}{1-\frac{(\epsilon-1)(1-\alpha)}{\epsilon}}\right)}_{>0} \frac{\widehat{w_t}}{p_t} = \frac{1}{1+\chi^{-1}} \left(\chi^{-1} - \frac{\frac{(\epsilon-1)(1-\alpha)}{\epsilon}}{1-\frac{(\epsilon-1)(1-\alpha)}{\epsilon}}\right) \widehat{C_{W,t}} \ ,$$

and is related to the output gap of the economy by:

$$\hat{Y}_t = \zeta \hat{Q}_t \ , \quad \text{where} \ \ \zeta \equiv \chi^{-1} \left(\chi^{-1} - \frac{\frac{(\epsilon-1)(1-\alpha)}{\epsilon}}{1-\frac{(\epsilon-1)(1-\alpha)}{\epsilon}}\right)^{-1} > 0 \ . \tag{B.1}$$

[2] See Table I.1 in the Appendix for a plausible calibration of the model parameters.

Proof of Lemma B.2. From $C_t = \rho A_t Q_t$, we obtain $\hat{C}_t = \hat{Q}_t$. We start from the flexible price economy's good market equilibrium condition, which can be written as

$$A_t \left(\frac{w_t^n}{p_t^n}\right)^{\frac{1}{\chi}} \frac{1}{A_t^{\frac{1}{\chi}}} = \rho A_t Q_t^n + \left(\frac{w_t^n}{p_t^n}\right)^{1+\frac{1}{\chi}} \frac{1}{A_t^{\frac{1}{\chi}}} , \tag{B.2}$$

where $\frac{w_t^n}{p_t^n}$ is the real wage in the flexible price economy. We subtract equation (B.2) from the analogous good market condition in the sticky price economy, and divide by $y_t^n \equiv A_t^{1-\frac{1}{\chi}}(\frac{w_t^n}{p_t^n})^{\frac{1}{\chi}}$, which yields the following result:

$$\underbrace{\frac{\left(\frac{w_t}{p_t}\right)^{\frac{1}{\chi}} - \left(\frac{w_t^n}{p_t^n}\right)^{\frac{1}{\chi}}}{\left(\frac{w_t^n}{p_t^n}\right)^{\frac{1}{\chi}}}}_{=\frac{1}{\chi}\widehat{\frac{w_t}{p_t}}} = \underbrace{\frac{C_t^n}{A_t^{1-\frac{1}{\chi}}\left(\frac{w_t^n}{p_t^n}\right)^{\frac{1}{\chi}}}}_{=1-\frac{(\epsilon-1)(1-\alpha)}{\epsilon}} \hat{C}_t + \underbrace{\frac{\left(\frac{w_t}{p_t}\right)^{1+\frac{1}{\chi}} - \left(\frac{w_t^n}{p_t^n}\right)^{1+\frac{1}{\chi}}}{A_t\left(\frac{w_t^n}{p_t^n}\right)^{\frac{1}{\chi}}}}_{=\frac{(\epsilon-1)(1-\alpha)}{\epsilon}\left(1+\frac{1}{\chi}\right)\widehat{\frac{w_t}{p_t}}} ,$$

which can be written as

$$\frac{1}{\chi}\widehat{\frac{w_t}{p_t}} = \left(1 - \frac{(\epsilon-1)(1-\alpha)}{\epsilon}\right)\hat{C}_t + \frac{(\epsilon-1)(1-\alpha)}{\epsilon}\underbrace{\left(1+\frac{1}{\chi}\right)\widehat{\frac{w_t}{p_t}}}_{=\hat{C}_{W,t}} ,$$

which, together with $\hat{C}_t = \hat{Q}_t$, leads to

$$\hat{Q}_t = \underbrace{\left(\chi^{-1} - \frac{\frac{(\epsilon-1)(1-\alpha)}{\epsilon}}{1-\frac{(\epsilon-1)(1-\alpha)}{\epsilon}}\right)}_{>0}\widehat{\frac{w_t}{p_t}} = \underbrace{\frac{1}{1+\chi^{-1}}\left(\chi^{-1} - \frac{\frac{(\epsilon-1)(1-\alpha)}{\epsilon}}{1-\frac{(\epsilon-1)(1-\alpha)}{\epsilon}}\right)}_{>0}\widehat{C_{W,t}} .$$

Finally, equation (B.1) follows by combining the previous expression with the market clearing condition $Y_t = C_t + C_{W,t}$, from which we obtain

$$\hat{Y}_t = \left(1 - \frac{(\epsilon-1)(1-\alpha)}{\epsilon}\right)\hat{Q}_t + \frac{(\epsilon-1)(1-\alpha)}{\epsilon}\hat{C}_{W,t} = \zeta\hat{Q}_t .$$

■

C Deriving the IS equation (10)

Proof of Proposition 1. With equations (2) with $\theta_t = 1$ and (6), capitalists' consumption C_t follows

$$\frac{dC_t}{C_t} = \left(i_t + (\sigma + \sigma_t^q)^2 - \rho\right) dt + (\sigma_t + \sigma_t^q) dZ_t. \tag{C.1}$$

where we use $i_t^m = i_t + (\sigma + \sigma_t^q)^2$. Thus, with equations (A.6), we obtain

$$\begin{aligned} d\hat{Q}_t = d\hat{C}_t &= \left(i_t - \underbrace{\left(r_t^n - \frac{(\sigma + \sigma_t^q)^2}{2} + \frac{\sigma^2}{2} \right)}_{\equiv r_t^T} \right) dt + \sigma_t^q dZ_t \\ &= \left(i_t - r_t^T\right) dt + \sigma_t^q dZ_t. \end{aligned} \tag{C.2}$$

Since we have risk-premium levels $\text{rp}_t = (\sigma_t + \sigma_t^q)^2$ in the sticky price economy and $\text{rp}_t^n = \sigma^2$ in the flexible price economy, we can express our risk-adjusted natural rate r_t^T as

$$r_t^T = r_t^n - \frac{1}{2}\left(\text{rp}_t - \text{rp}_t^n\right) = r_t^n - \frac{1}{2}\hat{rp}_t, \tag{C.3}$$

■

D Stochastic Stabilization in Section 4.3

Proof of Proposition 3. We derive the equilibrium when there is a Poisson (with ν as its parameter) probability that the economy returns to full stabilization after $\hat{T}^{\text{HOFG}}$. $\nu \in [0, +\infty)$, where $\nu = 0$ means no return to stabilization (as in Proposition 2). Central bank solves:

$$
\begin{aligned}
&\min_{\sigma_1^{q,L},\sigma_2^{q,L},\hat{T}^{\text{HOFG}}} \mathbb{E}_0 \int_0^{\hat{T}^{\text{HOFG}}} e^{-\rho t}\hat{Q}_t^2 dt + \mathbb{E}_0 \int_{\hat{T}^{\text{HOFG}}}^{\infty} e^{-\rho t}\cdot e^{-\nu\left(t-\hat{T}^{\text{HOFG}}\right)}\cdot \hat{Q}_t^2 dt, \\
&\text{s.t. } \begin{cases} d\hat{Q}_t = -(\underbrace{r_1^T(\sigma_1^{q,L})}_{<0})dt + (\sigma_1^{q,L})dZ_t, & \text{for } t < T, \\ d\hat{Q}_t = -(\underbrace{r_2^T(\sigma_2^{q,L})}_{>0})dt + (\sigma_2^{q,L})dZ_t, & \text{for } T \leq t < \hat{T}^{\text{HOFG}}, \\ d\hat{Q}_t = 0, & \text{for } t \geq \hat{T}^{\text{HOFG}}, \end{cases} , \\
&\text{with } \hat{Q}_0 = r_1^T(\sigma_1^{q,L})T + r_2^T(\sigma_2^{q,L})(\hat{T}^{\text{HOFG}} - T).
\end{aligned} \tag{D.1}
$$

where the discounting becomes $\rho + \nu > \rho$ after $\hat{T}^{\text{HOFG}}$, which is itself endogenous. The loss function in (D.1) can be written then as

$$
\begin{aligned}
&\min_{\sigma_1^{q,L},\sigma_2^{q,L},\hat{T}^{\text{HOFG}}} \mathbb{E}_0 \int_0^{\hat{T}^{\text{HOFG}}} e^{-\rho t}\hat{Q}_t^2 dt + \mathbb{E}_0 \int_{\hat{T}^{\text{HOFG}}}^{\infty} e^{-\rho t}\cdot e^{-\nu\left(t-\hat{T}^{\text{HOFG}}\right)}\cdot \hat{Q}_t^2 dt \\
&= \min_{\hat{T},\sigma_1^{q,L},\sigma_2^{q,L}} \underbrace{\int_0^{\hat{T}^{\text{HOFG}}} e^{-\rho t}\hat{Q}_{\text{d}}(t;\hat{T}^{\text{HOFG}})^2 dt}_{\text{From deterministic fluctuation}} + \underbrace{\left(\sigma_1^{q,L}\right)^2 \left[\frac{1-e^{-\rho T}}{\rho^2} - \frac{Te^{-\rho \hat{T}^{\text{HOFG}}}}{\rho}\left(\frac{\nu}{\rho+\nu}\right)\right]}_{\text{From stochastic fluctuation}} \\
&\quad + \underbrace{\left(\sigma_2^{q,L}\right)^2 \left[\left(\frac{e^{-\rho T} - e^{-\rho \hat{T}^{\text{HOFG}}}}{\rho^2}\right) - \left(\hat{T}^{\text{HOFG}} - T\right) e^{-\rho \hat{T}^{\text{HOFG}}} \frac{\nu}{\rho(\rho+\nu)}\right]}_{\text{From stochastic fluctuation}}
\end{aligned} \tag{D.2}
$$

where $\hat{Q}_d(t;\hat{T}^{\text{HOFG}})$ is defined in (II.2): we observe new terms appear compared with the baseline case of $\nu = 0$. Now, notice that if the central bank is allowed to maximize with respect to ν, then we obtain a corner solution with $\nu \to +\infty$. This means that the most efficient would be to immediately return to perfect stabilization, with a very small probability of no adjustment.

The central bank has control over $\sigma_1^{q,L}$, $\sigma_2^{q,L}$, and $\hat{T}^{\text{HOFG}}$, in addition to its conventional monetary policy tool $\{i_t\}$. We derive the first-order condition for $\hat{T}^{\text{HOFG}}$ as follows:

$$\begin{aligned} & 2 \cdot \underbrace{r_2^T(\sigma_1^{q,L})}_{>0} \int_0^{\hat{T}^{\text{HOFG}}} e^{-\rho t} \hat{Q}_{\text{d}}(t; \hat{T}^{\text{HOFG}}) dt + \underbrace{\left(\sigma_1^{q,L}\right)^2 e^{-\rho \hat{T}^{\text{HOFG}}} \left(\frac{\nu}{\rho+\nu}\right) T}_{>0} \\ & + \underbrace{\left(\sigma_2^{q,L}\right)^2 \left[\frac{e^{-\rho \hat{T}^{\text{HOFG}}}}{\rho+\nu} + \left(\hat{T}^{\text{HOFG}} - T\right) e^{-\rho \hat{T}^{\text{HOFG}}} \left(\frac{\nu}{\rho+\nu}\right)\right]}_{>0} = 0 \end{aligned} \tag{D.3}$$

from which we obtain

$$\int_0^{\infty} e^{-\rho t} \hat{Q}_{\text{d}}(t; \hat{T}^{\text{HOFG}}) dt = \int_0^{\hat{T}^{\text{HOFG}}} e^{-\rho t} \hat{Q}_{\text{d}}(t; \hat{T}^{\text{HOFG}} \| \sigma_1^{q,L} < 0, \sigma_2^{q,L} < 0) dt < 0 \; . \tag{D.4}$$

The first-order condition for $\hat{T}^{\text{HOFG}}$ indicates that, at the optimum, the central bank reduces the value of $\hat{T}^{\text{HOFG}}$ compared to $\hat{T}^{\text{TFG}}$ (traditional forward guidance). This is because when the central bank utilizes traditional forward guidance and achieves perfect stabilization for $t \geq \hat{T}^{\text{TFG}}$, the expression above becomes

$$\int_0^{\hat{T}^{\text{TFG}}} e^{-\rho t} \hat{Q}_{\text{d}}(t; \hat{T} \| \sigma_1^{q,L} = \sigma_1^{q,n} = 0, \sigma_2^{q,L} = \sigma_2^{q,n} = 0) dt = 0 \; , \tag{D.5}$$

which is derived by plugging $\sigma_1^{q,L} = 0$ and $\sigma_2^{q,L} = 0$ into equation (D.3).

Given that at the optimum, $\sigma_1^{q,L} < 0$ and $\sigma_2^{q,L} < 0$ (which we will demonstrate),

$$\hat{Q}_{\text{d}}(t; \hat{T}^{\text{HOFG}} \| \sigma_1^{q,L} = 0, \sigma_2^{q,L} = 0) < \hat{Q}_{\text{d}}(t; \hat{T}^{\text{HOFG}} \| \sigma_1^{q,L} < 0, \sigma_2^{q,L} < 0) \; .$$

Therefore, we deduce from equation (D.1) that at the optimum, $\hat{T}^{\text{HOFG}} < \hat{T}^{\text{TFG}}$, as evidenced by comparing (D.4) with (D.5).

To characterize the optimal values of $\sigma_1^{q,L}$ and $\sigma_2^{q,L}$, a **variational argument** is required. This is because $\sigma_1^{q,L}$ and $\sigma_2^{q,L}$ influence the levels of $r_1^T(\sigma_1^{q,L})$, $r_2^T(\sigma_2^{q,L})$, and $\hat{Q}_{\text{d}}(t; \hat{T}^{\text{HOFG}})$.

Specifically, we can derive:

$$\frac{\partial r_1^T(\sigma_1^{q,L})}{\partial \sigma_1^{q,L}} = -\left(\bar{\sigma} + \sigma_1^{q,L}\right) < 0, \quad \frac{\partial r_2^T(\sigma_2^{q,L})}{\partial \sigma_2^{q,L}} = -\left(\underline{\sigma} + \sigma_2^{q,L}\right) < 0 \ .$$

Determining $\sigma_1^{q,L}$ An increase in $\sigma_1^{q,L}$ leads to a decrease in $r_1^T(\sigma_1^{q,L})$, which alters the trajectory of $\hat{Q}_{\text{d}}(t; \hat{T}^{\text{HOFG}})$. This change is illustrated in Figure D.1, as depicted by the transition from the thick blue line to the dashed red line.

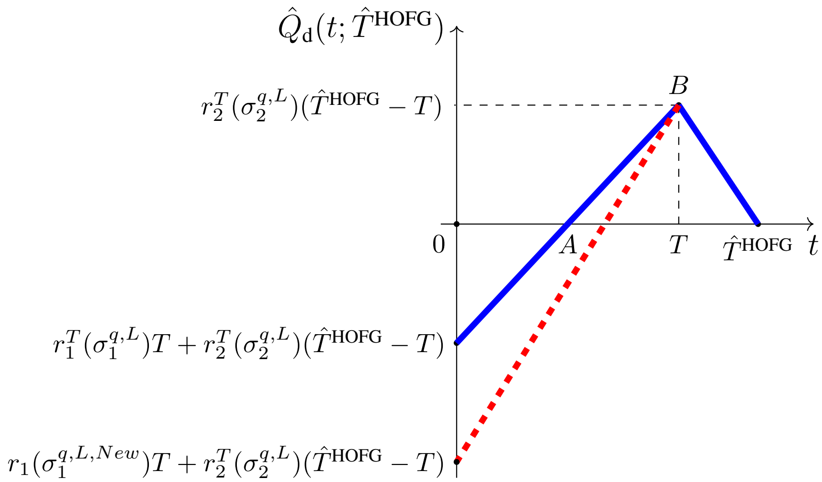

Figure D.1: Variation along $\sigma_1^{q,L}$. Increase to $\sigma_1^{q,L,New} > \sigma_1^{q,L}$.

Differentiating $\hat{Q}_{\text{d}}(t; \hat{T}^{\text{HOFG}}) = \int_t^{\hat{T}^{\text{HOFG}}} r_s^T ds$ with respect to $\sigma_1^{q,L}$, we obtain:

$$\frac{\partial \hat{Q}_{\text{d}}(t; \hat{T}^{\text{HOFG}})}{\partial \sigma_1^{q,L}} = \int_t^T -\left(\bar{\sigma} + \sigma_1^{q,L}\right) ds = -\left(\bar{\sigma} + \sigma_1^{q,L}\right)(T - t), \ \ \forall t \leq T \ .$$

To find optimal $\sigma_1^{q,L}$, we differentiate the objective function in (D.2) by $\sigma_1^{q,L}$ and obtain the following condition:

$$\left(\bar{\sigma} + \sigma_1^{q,L}\right) \int_0^T e^{-\rho t} \hat{Q}_{\text{d}}(t; \hat{T}^{\text{HOFG}})(T - t) dt = \left(\sigma_1^{q,L}\right) \left\{ \frac{1 - e^{-\rho T}}{\rho^2} - \frac{e^{-\rho \hat{T}^{\text{HOFG}}}}{\rho} \left[1 - \frac{\rho}{\rho + \nu}\right] \cdot T \right\} . \tag{D.6}$$

First, we obtain

$$\int_0^T e^{-\rho t}\hat{Q}_{\mathrm{d}}(t;\hat{T}^{\mathrm{HOFG}})(T-t)dt = \underbrace{\int_0^t e^{-\rho s}\hat{Q}_{\mathrm{d}}(s;\hat{T}^{\mathrm{HOFG}})ds\cdot(T-t)\Big|_0^T}_{=0} + \int_0^T \underbrace{\int_0^t e^{-\rho s}\hat{Q}_{\mathrm{d}}(s;\hat{T}^{\mathrm{HOFG}})ds}_{<0}\,dt < 0\,,$$

where $\int_0^t e^{-\rho s}\hat{Q}_{\mathrm{d}}(s;\hat{T}^{\mathrm{HOFG}})ds < 0$ for $t \leq T$, as derived in equation (D.4). Also, as we know

$$\begin{aligned}\frac{1-e^{-\rho T}}{\rho^2} - \frac{e^{-\rho\hat{T}^{\mathrm{HOFG}}}}{\rho}\left[1-\frac{\rho}{\rho+\nu}\right]T &\geq \frac{1-e^{-\rho T}}{\rho^2} - \frac{e^{-\rho\hat{T}^{\mathrm{HOFG}}}}{\rho}T \\ &= \underbrace{\int_0^T te^{-\rho t}dt}_{>0} + \underbrace{\frac{T}{\rho}e^{-\rho T} - \frac{T}{\rho}e^{-\rho\hat{T}^{\mathrm{HOFG}}}}_{\geq 0} > 0,\end{aligned} \tag{D.7}$$

from (D.6), we obtain that $\sigma_1^{q,L} < \sigma_1^{q,n} = 0$ at optimum.[3]

Determining $\sigma_2^{q,L}$ An increase in $\sigma_2^{q,L}$ leads to a decrease in $r_2^T(\sigma_2^{q,L})$, which alters the shape of $\hat{Q}_{\mathrm{d}}(t;\hat{T}^{\mathrm{HOFG}})$. This effect is illustrated in Figure D.2 by the transition from the thick blue line to the dashed red line. To further analyze this, we differentiate $\hat{Q}_{\mathrm{d}}(t;\hat{T}^{\mathrm{HOFG}})$ with respect to $\sigma_2^{q,L}$ and obtain:

$$\frac{\partial\hat{Q}_{\mathrm{d}}(t;\hat{T}^{\mathrm{HOFG}})}{\partial\sigma_2^{q,L}} = \begin{cases}\int_T^{\hat{T}^{\mathrm{HOFG}}} -\left(\underline{\sigma}+\sigma_2^{q,L}\right)ds = -\left(\underline{\sigma}+\sigma_2^{q,L}\right)(\hat{T}^{\mathrm{HOFG}}-T)\,, & t<T\,,\\ \int_t^{\hat{T}^{\mathrm{HOFG}}} -\left(\underline{\sigma}+\sigma_2^{q,L}\right)ds = -\left(\underline{\sigma}+\sigma_2^{q,L}\right)(\hat{T}^{\mathrm{HOFG}}-t)\,, & T\leq t\leq\hat{T}^{\mathrm{HOFG}}\,.\end{cases}$$

[3]Note that in (D.6), due to the additional term

$$\frac{e^{-\rho\hat{T}^{\mathrm{HOFG}}}}{\rho}\left[1-\frac{\rho}{\rho+\nu}\right]\cdot T,$$

$\sigma_1^{q,L}$ becomes more negative at optimum taking $\hat{T}^{\mathrm{HOFG}}$ and $\sigma_2^{q,L}$ as given, compared with our benchmark case in which $\nu = 0$.

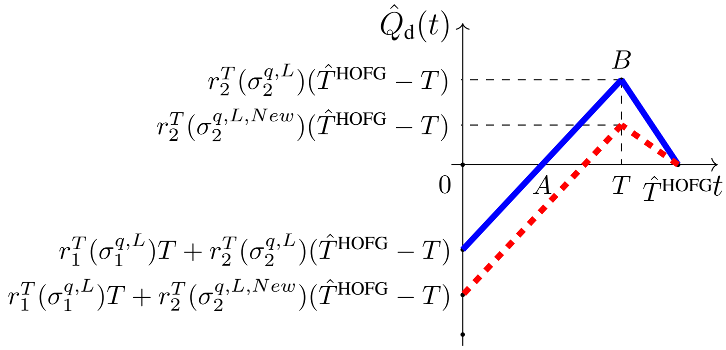

Figure D.2: Variation along $\sigma_2^{q,L}$. Increase to $\sigma_2^{q,L,New} > \sigma_2^{q,L}$.

To find the optimal $\sigma_2^{q,L}$, we differentiate the objective function in (D.2) by $\sigma_2^{q,L}$ and obtain

$$\left(\underline{\sigma}+\sigma_2^{q,L}\right)\left(\int_0^T e^{-\rho t}\hat{Q}_{\rm d}(t;\hat{T}^{\rm HOFG})(\hat{T}^{\rm HOFG}-T)dt+\int_T^{\hat{T}^{\rm HOFG}} e^{-\rho t}\underbrace{\hat{Q}_{\rm d}(t;\hat{T}^{\rm HOFG})}_{>0}(\hat{T}^{\rm HOFG}-t)dt\right)$$
$$=(\sigma_2^{q,L})\left\{\frac{e^{-\rho T}-e^{-\rho\hat{T}}}{\rho^2}-\frac{e^{-\rho\hat{T}^{\rm HOFG}}}{\rho}\left[1-\frac{\rho}{\rho+\nu}\right]\left(\hat{T}^{\rm HOFG}-T\right)\right\}, \tag{D.8}$$

from which we can demonstrate that at the optimum, $\sigma_2^{q,L}<0$ must be satisfied, given that

$$\int_0^T e^{-\rho t}\hat{Q}_{\rm d}(t;\hat{T}^{\rm HOFG})(\hat{T}^{\rm HOFG}-T)dt+\int_T^{\hat{T}^{\rm HOFG}} e^{-\rho t}\underbrace{\hat{Q}_{\rm d}(t;\hat{T}^{\rm HOFG})}_{>0}(\hat{T}^{\rm HOFG}-t)dt$$
$$<\int_0^T e^{-\rho t}\hat{Q}_{\rm d}(t;\hat{T}^{\rm HOFG})(\hat{T}^{\rm HOFG}-T)dt+\int_T^{\hat{T}^{\rm HOFG}} e^{-\rho t}\underbrace{\hat{Q}_{\rm d}(t;\hat{T}^{\rm HOFG})}_{>0}(\hat{T}^{\rm HOFG}-T)dt$$
$$=(\hat{T}^{\rm HOFG}-T)\underbrace{\int_0^{\hat{T}^{\rm HOFG}} e^{-\rho t}\hat{Q}_{\rm d}(t;\hat{T}^{\rm HOFG})dt}_{<0}<0\ ,$$

where the final inequality is derived from equation (D.4), and

$$\frac{e^{-\rho T}-e^{-\rho\hat{T}}}{\rho^2}-\frac{e^{-\rho\hat{T}^{\rm HOFG}}}{\rho}\left[1-\frac{\rho}{\rho+\nu}\right]\left(\hat{T}^{\rm HOFG}-T\right)\geq\frac{e^{-\rho T}-e^{-\rho\hat{T}}}{\rho^2}-\frac{e^{-\rho\hat{T}^{\rm HOFG}}}{\rho}\left(\hat{T}^{\rm HOFG}-T\right)$$
$$=\int_T^{\hat{T}^{\rm HOFG}} e^{-\rho t}(t-T)dt>0. \tag{D.9}$$

Equation (D.8) proves that $\sigma_2^{q,L} < 0$ at optimum.[4] Therefore, we have proven that during periods of high TFP volatility (i.e., $t < T$) and low TFP volatility with forward guidance (i.e., $T \leq t \leq \hat{T}^{\text{HOFG}}$), a central bank aims to target financial volatility levels below those in a flexible price economy: $\sigma_1^{q,L} < \sigma_1^{q,n} = 0$ and $\sigma_2^{q,L} < \sigma_2^{q,n} = 0$. Such intervention reduces the required risk premium and raises the asset price level $\hat{Q}_t$, thereby increasing output.

■

Proof of Corollary 1. Note that $\nu = \infty$ implies that full stabilization immediately follows after $\hat{T}^{\text{HOFG}}$ when the zero policy rate regime is over. It corresponds to the traditional forward guidance case of Section 4.1, so when $\nu = \infty$, the only feasible $\left(\sigma_1^{q,L}, \sigma_2^{q,L}, \hat{T}^{\text{HOFG}}\right)$ would be $(0, 0, \hat{T})$ in this case. Since for every ν, $\left(\sigma_1^{q,L}, \sigma_2^{q,L}, \hat{T}^{\text{HOFG}}\right) = (0, 0, \hat{T}^{\text{TFG}})$ is feasible, we obtain

$$\lim_{\nu \to +\infty^-} \mathbb{L}^{Q,*}\left(\{\hat{Q}_t\}_{t \geq 0}, \nu\right) \leq \mathbb{L}^{Q,*}\left(\{\hat{Q}_t\}_{t \geq 0}, \nu = \infty\right) .$$

To obtain the strict inequality between the two sides, we compare the first-order conditions for $\hat{T}^{\text{HOFG}}$ when $\nu = \infty$ and $\nu \to \infty$. When $\nu = \infty$, the optimality is given by (15), which can be written as

$$\int_0^{\hat{T}^{\text{HOFG}}} e^{-\rho t} \hat{Q}_{\text{d}}(t; \hat{T}^{\text{HOFG}}) dt = 0 , \tag{D.10}$$

where $\hat{Q}_{\text{d}}$ is defined in (II.2). In contrast, when $\nu \to \infty$, the first-order condition of $\hat{T}^{\text{HOFG}}$ in (D.3) becomes

$$2 \cdot \underbrace{r_2^T(\sigma_1^{q,L})}_{>0} \int_0^{\hat{T}^{\text{HOFG}}} e^{-\rho t} \hat{Q}_{\text{d}}(t; \hat{T}^{\text{HOFG}}) dt + \underbrace{\left(\sigma_1^{q,L}\right)^2 e^{-\rho \hat{T}^{\text{HOFG}}} T}_{>0} + \underbrace{\left(\sigma_2^{q,L}\right)^2 \left[\left(\hat{T}^{\text{HOFG}} - T\right) e^{-\rho \hat{T}^{\text{HOFG}}}\right]}_{>0} = 0$$

[4] Note that in (D.8), due to the additional term

$$\frac{e^{-\rho \hat{T}^{\text{HOFG}}}}{\rho} \left[1 - \frac{\rho}{\rho + \nu}\right] \cdot (\hat{T}^{\text{HOFG}} - T),$$

$\sigma_2^{q,L}$ becomes more negative at optimum taking $\hat{T}^{\text{HOFG}}$ and $\sigma_1^{q,L}$ as given, compared with our benchmark case in which $\nu = 0$.

which is different from the above (D.10). Therefore, we obtain

$$\lim_{\nu \to +\infty^-} \mathbb{L}^{Q,*}\left(\{\hat{Q}_t\}_{t\geq 0}, \nu\right) < \mathbb{L}^{Q,*}\left(\{\hat{Q}_t\}_{t\geq 0}, \nu = \infty\right) .$$

∎

E Stochastic T in Section 3 and Section 4.1

Here we prove the result of Section 3 and Section 4.1 that $\sigma_t^q = \sigma_t^{q,n} \equiv 0$ still holds even when the mandatory ZLB duration T is stochastic. First, we do not consider the traditional forward guidance policy.

For illustration purposes, we assume that T follows a discrete distribution: T_1, T_2, and T_3 with probabilities p_1, p_2, and p_3 with $p_1 + p_2 + p_3 = 1$. The same logic can be applied to more general cases where T has a continuous distribution. We keep assuming that after T is realized, i.e., the ZLB ends, the monetary authority achieves perfect stabilization based on a rule in (11). We similarly rely on the backward induction. First, we know certainly that after T_3, the economy is fully stabilized, implying $\sigma_t^q = 0$ for $t \geq T_3$. For $t \in [T_2, T_3)$,

1. If the ZLB already ended at T_1 or T_2, then $\sigma_t^q = 0$.

2. The ZLB has not ended: then it is certain that $T = T_3$ and $\hat{Q}_t = 0$ for $t \geq T_3$, which means that $\sigma_t^q = 0$ for $t \in (T_2, T_3)$. In that case, $\hat{Q}_{T_2} = \underline{r}(T_3 - T_2) < 0$ is determined.

For $t \in [T_1, T_2)$, we know that

1. If the ZLB already ended at T_1, then $\sigma_t^q = 0$.

2. The ZLB has not ended: then it is for sure that $T = T_2$ or $T = T_3$. At $t = T_2 - dt$ for small $dt > 0$, $\hat{Q}_{T_2 - dt}$ is determined by a conditional probability-weighted linear combination of 0 (when $T = T_2$) and $\underline{r}(T_3 - T_2)$ (when $T = T_3$), so that

$$\hat{Q}_{T_2 - dt} = \underline{r}dt + \frac{p_2}{1 - p_1} \cdot 0 + \frac{p_3}{1 - p_1} \cdot \underline{r}(T_3 - T_2).$$

Since $\hat{Q}_{T_2-dt}$ is determined, $\sigma_t^q = 0$ for $t \in [T_1, T_2)$.

For $t < T_1$, we know that

1. $T = T_1$ or T_2 or T_3. At $t = T_1 - dt$ for small $dt > 0$, $\hat{Q}_{T_1-dt}$ is determined by a conditional probability-weighted linear combination of 0 (when $T = T_1$), $\underline{r}(T_2 - T_1)$ (when $T = T_2$) and $\underline{r}(T_3 - T_1)$ (when $T = T_3$), so that

$$\hat{Q}_{T_1-dt} = \underline{r}dt + p_1 \cdot 0 + p_2 \cdot \underline{r}(T_2 - T_1) + p_3 \cdot \underline{r}(T_3 - T_1).$$

   Since $\hat{Q}_{T_1-dt}$ is determined, $\sigma_t^q = 0$ for $t < T_1$.

Therefore, $\sigma_t^q = \sigma_t^{q,n} = 0$ for all t even if ZLB duration T is stochastic.

Traditional forward guidance When T is stochastic, the zero rate duration under traditional forward guidance, i.e., $\hat{T}$ in Section 4.1, becomes stochastic as well and dependent on T. The above logic can be applied in this case, and we can similarly prove that if the monetary authority commits to perfectly stabilizing the economy after any realized $\hat{T}$, then $\sigma_t^q = \sigma_t^{q,n} = 0$ for $t \leq \hat{T}$.

References

Baxter, Marianne and Robert King, "Productive externalities and business cycles," *Working Paper*, 1991.