Self-fulfilling Volatility and a New Monetary Policy[*,†]

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Abstract

This paper demonstrates that in macroeconomic models with nominal rigidities, a global solution exists, supporting an alternative equilibrium where traditional Taylor rules give rise to self-fulfilling aggregate volatility. Within the rational expectations framework, we establish that individually optimal, path-dependent consumption strategies can generate endogenous volatility in a self-fulfilling manner, propelling the entire economy into crises (booms) characterized by elevated (reduced) aggregate risk. This outcome stems from the inability of traditional policy rules to target the expected growth rate of aggregate output, which comprises not only the policy rate but also the precautionary savings channel; the latter ultimately determining the degree of households' intertemporal substitution. We then propose a new policy rule that targets both conventional mandates and aggregate volatility, and outline the necessary conditions to reestablish determinacy and attain full stabilization.

Keywords: Taylor Rules, Self-fulfilling Volatility, Risk-Premium

JEL codes: E32, E43, E44, E52

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1 Introduction

How should monetary policy respond to fluctuations in aggregate market volatility? The prevailing perspective suggests that central banks require two distinct sets of instruments: macroprudential policies to preserve the stability of markets, that is, maintaining a stable level of market volatility, and monetary and fiscal policies to achieve the conventional goal of macroeconomic stabilization. Nevertheless, the debate surrounding this matter remains unresolved for numerous reasons. For instance, aggregate volatility is inherently endogenous, and integrating its influence into macroeconomic models presents a significant challenge. Mainstream macroeconomic frameworks often rely on approximation techniques that simplify or entirely disregard higher-order terms associated with economic volatility. Alternatively, they depend on numerical solution methods that may obscure the underlying economic intuition.

In this paper, we demonstrate that within a macroeconomic model featuring nominal rigidities, Taylor rules, irrespective of their responsiveness to typical business cycle mandates (e.g., output gap), permit aggregate volatility to emerge in a self-fulfilling manner. We illustrate this insight within two macroeconomic models: (i) the standard New-Keynesian model,[1] and (ii) a model incorporating stock markets and portfolio decisions, the latter of which is provided in Online Appendix B. Our continuous-time characterization of the problem allows the models' solutions to remain tractable, yielding closed-form expressions for the time-varying aggregate volatility and business cycle variables, all of which are endogenously determined.

In the standard New-Keynesian model, the economy's time-varying aggregate volatility has a first-order impact on aggregate consumption demand through the precautionary savings channel. More specifically, heightened aggregate volatility leads households to increase their precautionary savings, reducing aggregate demand and output, while the aggregate volatility itself is determined by fluctuations in output. In this setting, households can generate aggregate volatility through their intertemporal consumption coordination under rational expectations. For instance, consider a scenario where households at time 0 suddenly believe that the economy in the next period will be more volatile. They decrease their current consumption and increase precautionary savings, resulting in a recession at time 0. In period 1, the initial fear at time 0 regarding the volatility of the time 1 economy must be validated. This can be achieved if, for each possible realized consumption at period

[1] See, for example, Galí (2015).

1, there exists a corresponding conditional volatility of period 2 consumption. Specifically, a higher realization of time 1 consumption should be accompanied by a lower conditional volatility of period 2 consumption, leading to a decreased degree of precautionary savings. Essentially, the household's belief in the current volatility is shaped by their expectations in the previous period and justified by their actions in future periods. Note that our equilibrium construction with self-generated volatility is made possible due to nominal rigidities: the path-dependent consumption strategy of households determines the stochastic output paths, as the economy is driven by demand.

In the specific rational expectations equilibrium we refer to as the "martingale" equilibrium, the economy (i.e., output gap) adheres to a martingale, meaning that, on average, the next period economy remains at the current level. As the conditional volatility of the subsequent period's consumption declines as the economy approaches the stabilized path (i.e., the flexible price economy), the stabilized path functions as an attractor for all sample paths. Consequently, after generating a self-fulfilling volatility shock, the economy is almost certainly stabilized in the long run. However, on the equilibrium path, and until the economy is nearly stabilized following the emergence of the initial volatility in a self-fulfilling way, it experiences a prolonged recession accompanied by increased aggregate volatility. We demonstrate that a *probability-zero event*, in which the self-created conditional volatility ultimately diverges toward infinity, enables the initial appearance of self-fulfilling volatility and ensures that the economy follows a martingale, even if it is almost surely stabilized in the long run. We relate this property to an endogenously generated rare-disaster event that arises in a self-fulfilling manner.

The traditional Taylor rules' failure to prevent self-fulfilling volatility arises from their neglect or inability to directly address it. Consequently, we propose a novel policy rule that targets aggregate volatility in a specific way, effectively preventing the onset of a self-fulfilling volatility shock and ensuring determinacy. This approach suggests that the rate influencing households' intertemporal substitution should reflect the pressures of precautionary savings due to aggregate volatility, alongside the policy rate. Therefore, to ensure determinacy and stabilization, it is necessary the expected growth rate of consumption (or output), not solely the policy rate, targets business cycle mandates.

Second model To facilitate a clearer understanding of the problem, we introduce a second macroeconomic model in Online Appendix B incorporating stock markets and portfolio decisions, wherein aggregate volatility is associated with financial instability and reflected in the financial market risk premium. This model showcases a similar role for aggregate stock price volatility and risk premium: a more volatile financial market with a higher risk premium reduces aggregate financial wealth through individual investors' portfolio decisions, subsequently diminishing aggregate demand and output. Due to the analogous mathematical structure concerning the influence of aggregate volatility on aggregate demand, we can construct an equilibrium in which aggregate stock price volatility is generated in a self-fulfilling manner and merely reflects the volatility of the underlying firms. The possibility of self-fulfilling volatility in this specific context can also be interpreted as follows: the fear of a financial crisis resulting from an increase in risk premium and stock market volatility renders investors less inclined to invest in the stock market, lowering current asset prices and wealth, thereby producing self-fulfilling increases in the expected stock market return and risk premium.

There we argue for a *generalized* Taylor rule that targets risk-premium, which achieves what we call ultra-divine coincidence: the simultaneous stabilization of inflation, output gap, and risk-premium (equivalently, aggregate stock price volatility). In addition, our calibrated model in Online Appendix B quantitatively matches with impulse-response functions we obtain from our structural vector autoregrssion exercise of Online Appendix C.

Related literature Our model with stock markets shares similarities with Caballero and Simsek (2020a,b) in terms of incorporating an endogenous asset market interwoven with the fluctuations of the business cycle. However, while their framework focuses on how behavioral biases can generate intriguing crisis dynamics through the feedback loop between asset markets and business cycles,[2] our attention centers on the traditional policy rule under rational expectations and the existence of alternative equilibria arising from higher-order moments.

While Benhabib et al. (2002) study monetary-fiscal regimes in regards to eliminating indeterminacy issues posed by the ZLB and Obstfeld and Rogoff (2021) show how a probabilistic (and small) fiscal currency backing rules out speculative hyper-inflation in monetary models, our focus is on aggregate volatility's self-fulfilling apparition outside the ZLB, as well as alternative monetary policy rules.

Our equilibrium determinacy results resemble those of Acharya and Dogra (2020) and

[2] Caballero and Simsek (2020b) present a model with optimists and pessimists who hold differing beliefs about the probability of an imminent recession or boom. During zero lower bound (ZLB) episodes, an endogenous decline in risky asset valuation, triggered by a reduction in optimists' wealth, leads to a demand recession. We explore related ZLB issues in a separate paper, Dordal i Carreras and Lee (2024).

Khorrami and Mendo (2022). While Acharya and Dogra (2020) investigates how determinacy conditions change in the presence of exogenous idiosyncratic risks that are functions of aggregate output, we explore the existence of self-fulfilling aggregate volatility and examine the monetary policy that restores determinacy. Khorrami and Mendo (2022) study similar equilibrium indeterminacy issues around the aggregate volatility at the ZLB, whereas we focus on cases outside the ZLB and show how exactly we can construct an equilibrium that supports self-fulfilling volatility.

Online Appendix Online Appendix A offers a detailed account of the equilibrium conditions in Section 2. Online Appendix B provides a model with stock markets, and Online Appendix C present evidence illustrating the significance of financial volatility as a driver of business cycle fluctuations, employing a structural Vector Autoregression approach. Online Appendix D provides additional figures and tables. Finally, Online Appendix E contains derivations and proofs for Online Appendix B.

2 Standard Non-linear New Keynesian Model

This section illustrates that a *non-linear* characterization of the equilibrium enables higher-order moments tied to the aggregate business cycle volatility to have a first-order impact on the business cycle dynamics. This feature will have important implications for equilibrium determinacy and the proper management of monetary policy needed to stabilize the business cycle. More detailed characterization of optimality conditions are provided in Online Appendix A.

The representative household owns the firms of this economy and receives their profits via lump-sum transfers. For simplicity, we assume a perfectly rigid price level: $p_t = \bar{p}$, $\forall t$ so there is no inflation in the economy. This assumption is not crucial but allows us to focus on the key mechanism we want to illustrate.[3] The optimization problem of the household is given by

$$\max_{\{B_t, C_t, L_t\}_{t \geq 0}} \mathbb{E}_0 \int_0^\infty e^{-\rho t} \left[\log C_t - \frac{L_s^{1+\frac{1}{\eta}}}{1+\frac{1}{\eta}} \right] dt \quad , \quad \text{s.t. } \dot{B}_t = i_t B_t - \bar{p} C_t + w_t L_t + D_t, \quad (1)$$

where C_t and L_t are her consumption and labor supply, respectively, η is the Frisch elastic-

[3] Online Appendix B relaxes this assumption and introduces price stickiness à la Calvo (1983).

ity of labor supply, B_t is her nominal holding of bonds, and D_t are the entire firms' profits and fiscal transfers from the government. w_t is the equilibrium wage, and i_t is the policy rate set by the central bank. We assume that there is no government spending, and therefore aggregate consumption determines production in this environment with price rigidity. For simplicity, the bond market is in zero net supply in equilibrium. Finally, ρ is the time discount rate.

We obtain

$$-i_t dt = \mathbb{E}_t\left(\frac{d\xi_t^N}{\xi_t^N}\right), \text{ where } \xi_t^N = e^{-\rho t}\frac{1}{\bar{p}}\frac{1}{C_t}, \tag{2}$$

as the intertemporal optimality condition of problem (1), where $\frac{d\xi_t^N}{\xi_t^N}$ is the instantaneous (nominal) stochastic discount factor, and its expected value equals the (minus) nominal risk-free rate $-i_t dt$.[4] Due to the rigid price assumption, there is no inflation, i.e., $\pi_t = 0,\ \forall t$, thereby the real and nominal risk-free rates of the economy are equal, i.e., $r_t = i_t$, where r_t is the real interest rate.

We can rewrite equation (2) as

$$\mathbb{E}_t\left(\frac{dC_t}{C_t}\right) = (i_t - \rho)dt + \mathrm{Var}_t\left(\frac{dC_t}{C_t}\right), \tag{3}$$

where the last term $\mathrm{Var}_t(\frac{dC_t}{C_t})$ arises from the endogenous volatility of the aggregate consumption process. Note that this term is usually a second-order term and therefore is typically dropped out in log-linearized models. In contrast, our non-linear characterization (3) properly accounts for consumption volatility and allows it to affect the drift of the aggregate consumption process, where the volatility as well as the drift is an endogenous object. This additional term reflects the usual *precautionary savings channel*, in which a more volatile business cycle leads to an increased demand for riskless savings, which in turn leads to a drop in current consumption and a higher expected growth for the consumption process.

An individual firm i produces with the linear production function: $Y_t^i = A_t L_t^i$ where L_t^i is firm i's labor hiring, and A_t is the economy's total factor productivity assumed to be exogenous and to follow a geometric Brownian motion with drift:

$$\frac{dA_t}{A_t} = g dt + \sigma dZ_t, \tag{4}$$

where g is its expected growth rate and σ is what we call 'fundamental' volatility, assumed

[4] Online Appendix A provides the Hamilton-Jacobi-Bellman (HJB) equation-based derivation for (2).

to be constant over time.[5] It follows that firms' profits to be rebated can be written as $D_t = \bar{p}Y_t - w_t L_t$. We assume that all the aggregate variables are adapted to the filtration $(\mathcal{F}_t)_{t\in\mathbb{R}}$ generated by the process in (4) in a given *filtered* probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\in\mathbb{R}}, \mathbb{P})$.

Flexible price equilibrium as benchmark With the usual Dixit-Stiglitz monopolistic competition among firms, we can characterize the flexible price equilibrium where firms can freely choose their prices, in contrast to the fully rigid price, i.e., $p_t = \bar{p}$. The flexible price equilibrium outcomes are called 'natural' as central banks in the presence of price rigidity target these outcomes with their monetary tools. As we prove in Online Appendix A.2.2, the natural output Y_t^n follows

$$\frac{dY_t^n}{Y_t^n} = \Big(\underbrace{r^n}_{\text{Natural rate}} - \rho + \sigma^2\Big)\, dt + \underbrace{\sigma}_{\text{Natural volatility}} dZ_t, \tag{5}$$

where $r^n = \rho + g - \sigma^2$ is defined as the natural interest rate. From the monetary authority's perspective, the process in (5) is an exogenous process that monetary policy cannot affect nor control. Note that natural output Y_t^n follows a geometric Brownian motion with the volatility σ, which equals the volatility of A_t process in (4).

Rigid price equilibrium and the 'gap' economy Going back to the 'rigid' price economy, we first introduce σ_t^s as the *excess* volatility of the growth rate of the output process $\{Y_t\}$, compared with the benchmark flexible price economy output in (5). Then:

$$\mathrm{Var}_t\left(\frac{dY_t}{Y_t}\right) = (\sigma + \sigma_t^s)^2 dt \tag{6}$$

holds by definition. Note that σ_t^s is an 'endogenous' volatility term to be determined in equilibrium. By plugging equation (6) into the nonlinear Euler equation (3), we obtain

$$\frac{dY_t}{Y_t} = \left(i_t - \rho + (\sigma + \sigma_t^s)^2\right) dt + (\sigma + \sigma_t^s) dZ_t. \tag{7}$$

[5] This assumption is made for simplicity and our analysis can be extended to include cases where σ_t is time-varying and adapted to the Brownian motion Z_t.

With the usual definition of output gap $\hat{Y}_t = \ln\left(\frac{Y_t}{Y_t^n}\right)$, we obtain[6]

$$d\hat{Y}_t = \left(i_t - \left(r^n \overbrace{-\frac{1}{2}(\sigma + \sigma_t^s)^2 + \frac{1}{2}\sigma^2}^{\text{New terms}} \right) \right) dt + \sigma_t^s dZ_t, \tag{8}$$

which features an interesting feedback effect that is omitted in log-linearized equations:[7] given the policy rate i_t, a rise in the endogenous volatility σ_t^s pushes up the drift of (8) and lowers output gap $\hat{Y}_t$. The intuition follows from the households' precautionary behavior we see in (3): households respond to a higher economic volatility with increased savings and lower consumption, thereby inducing a recession.

Define the *risk-adjusted* natural rate as

$$r_t^T = r^n - \frac{1}{2}(\sigma + \sigma_t^s)^2 + \frac{1}{2}\sigma^2. \tag{9}$$

and note that r_t^T is itself endogenous: it negatively depends on the endogenous aggregate (excess) volatility σ_t^s. This risk-adjusted natural rate can be regarded as a new reference risk-free rate of the economy at which i_t completely eliminates the drift of the output gap.

2.1 Taylor rules and Indeterminacy

Now we study the conventional Taylor rule and its capacity to guarantee model determinacy and economic stabilization. We assume that the central bank sets the risk-free rate i_t of the economy according to:

$$i_t = r^n + \phi_y \hat{Y}_t, \ \text{ where } \ \phi_y > 0. \tag{10}$$

Condition $\phi_y > 0$ is the 'Taylor principle' that guarantees unique equilibrium in conventional log-linearized models that omit the first-order effects of aggregate volatility. Here, we ask whether the policy in (10) retains the capacity to determine a unique equilibrium in our non-linear economy that features the feedback relationship between output gap volatil-

[6] In (7), we assume that the current output Y_t is adapted to the filtration $(\mathcal{F}_t)_{t\in\mathbb{R}}$ generated by the technology process in (4). Therefore, σ_t^s in (7) can be interpreted as a *fundamental* excess volatility.

[7] For illustrative purposes, compare (8) with the conventional IS equation given by $d\hat{Y}_t = (i_t - r^n)\,dt + \sigma_t^s dZ_t$ where the endogenous aggregate volatility σ_t^s has no first-order effect on the drift.

ity and its drift explained in (8). Plugging equation (10) into equation (8), we obtain

$$d\hat{Y}_t = \left(\phi_y \hat{Y}_t - \frac{\sigma^2}{2} + \frac{(\sigma + \sigma_t^s)^2}{2}\right) dt + \sigma_t^s dZ_t \tag{11}$$

as the dynamics for output gap $\hat{Y}_t$.

Multiple equilibria Omitting the new volatility terms from the drift of (11), we obtain the usual log-linearized version of the $\hat{Y}_t$ dynamics as

$$d\hat{Y}_t = \left(\phi_y \hat{Y}_t\right) dt + \sigma_t^s dZ_t. \tag{12}$$

With the dynamics described by (12), Blanchard and Kahn (1980) proves the existence of a *unique* rational expectations equilibrium when the Taylor principle $\phi_y > 0$ is satisfied: $\hat{Y}_t = 0$, $\forall t$, which corresponds to a fully stabilized economy.

We now claim that this result does not hold in the current $\hat{Y}_t$ process in (11), and there are a variety of rational expectations equilibria consistent with (10). In particular, the feedback effect from the endogenous volatility σ_t^s of the output gap to its drift in equation (11) enables the appearance of *self-fulfilling* volatility σ_t^s. Our objective here is to provide a rational expectations equilibrium that supports the apparition of an initial excess volatility $\sigma_0^s > 0$, by constructing directly an equilibrium path where the $\hat{Y}_t$ follows a martingale.[8] The case of negative volatility (i.e., $\sigma_0^s < 0$) can be similarly constructed. Our martingale equilibrium construction (i) supports an initial volatility $\sigma_0^s > 0$, i.e., explain why $\sigma_0^s > 0$ can arise in a self-fulfilling way, and (ii) does not diverge on expectation in the long-run, consistent with the traditional definition of a rational expectations equilibrium (see e.g., Blanchard and Kahn (1980)).[9]

Martingale equilibrium We provide the explicit equilibrium in which an initial volatility $\sigma_0^s > 0$ appears and $\hat{Y}_t$ is a martingale, consistent with the dynamics in (11). First, the

[8]Our martingale equilibrium is one of possible fundamental equilibria consistent with (10). Its construction, however, illustrates how a sudden rise in endogenous volatility interacts with monetary policy and drives business cycles.

[9]The apparition of the initial volatility σ_0^s is not in the economy's filtration $(\mathcal{F}_t)_{t\in\mathbb{R}}$. This can be regarded as a sunspot shock to the excess volatility σ_t^s, while aggregate variables jump in response to its appearance.

$\{\hat{Y}_t\}$ process' drift must be zero in order for it to become martingale, which gives:

$$\hat{Y}_t = -\frac{(\sigma + \sigma_t^s)^2}{2\phi_y} + \frac{\sigma^2}{2\phi_y}. \tag{13}$$

The martingale equilibrium guarantees the rationality of the equilibrium, as on average the path of $\{\hat{Y}_t\}$ stays at the same level (thereby does not diverge in the long run), satisfying $\mathbb{E}_0(\hat{Y}_t) = \hat{Y}_0$ (convergence in expectations in Blanchard and Kahn (1980)). The last step is to show the existence of a stochastic path for $\{\sigma_t^s\}$ starting from σ_0^s that supports this equilibrium. Using (11) and (13), we obtain that σ_t^s starting from σ_0^s follows[10]

$$d\sigma_t^s = -(\phi_y)^2 \frac{(\sigma_t^s)^2}{2(\sigma + \sigma_t^s)^3} dt - \phi_y \frac{\sigma_t^s}{\sigma + \sigma_t^s} dZ_t. \tag{14}$$

Therefore, equations (13) and (14) constitute the dynamics of our constructed rational expectations equilibrium supporting self-fulfilling volatility $\sigma_0^s > 0$. The following Proposition 1 sheds lights on the behavior of $\{\hat{Y}_t, \sigma_t^s\}$ under the martingale equilibrium and finds that: even if the economy is hit by an initial self-fulfilling volatility shock $\sigma_0^s > 0$, the business cycle almost surely converges to the perfectly stabilized path in the long run through monetary stabilization based on Taylor rules. Nonetheless, a few paths that occur with *tiny* probability do not converge and explode asymptotically, sustaining the initial volatility $\sigma_0^s > 0$ due to the forward-looking nature of the economy.

Proposition 1 (Taylor Rules and Indeterminacy) *For any value of $\phi_y > 0$:*

1. Indeterminacy*: there is always a rational expectations equilibrium (REE) that supports initial volatility $\sigma_0^s > 0$ and is represented by $\hat{Y}_t$ dynamics in equation* (13)*, and σ_t^s process in equation* (14).

2. Properties*: the equilibrium that supports an initial volatility $\sigma_0^s > 0$ satisfies:*

$$(i)\ \sigma_t^s \overset{a.s}{\to} \sigma_\infty^s = 0,\ \ (ii)\ \hat{Y}_t \overset{a.s}{\to} 0,\ \ \textit{and}\ \ (iii)\ \mathbb{E}_0\left(\max_t (\sigma_t^s)^2\right) = \infty.$$

[10]When $\sigma = 0$, $\forall t$, equation (14) becomes the following Bessel process:

$$d\sigma_t^s = -\frac{(\phi_y)^2}{2\sigma_t^s} dt - \phi_y dZ_t,$$

which stops when σ_t^s reaches zero. For general properties of Bessel processes, see Lawler (2019).

The results that $\sigma_t^s \overset{a.s}{\to} \sigma_\infty^s = 0$ and $\hat{Y}_t \overset{a.s}{\to} 0$ imply that the equilibrium paths starting from an initial volatility $\sigma_0^s > 0$ are almost surely stabilized in the long run. Still, almost sure stabilization of paths is compatible with a self-fulfilling appearance of $\sigma_0^s > 0$ by the latter result of the Proposition, $\mathbb{E}_0(\max_t(\sigma_t^s)^2) = \infty$, which implies that an initial self-fulfilling shock in σ_0^s is sustained by a *vanishing* probability of an ∞-large equilibrium volatility in some future paths. Still, we have $\lim_{t\to\infty} |\mathbb{E}_0(\hat{Y}_t)| = |\hat{Y}_0| < \infty$, satisfying the 'convergence in expectation' criteria in Blanchard and Kahn (1980).

Intuition Here we explain in a detailed manner the intuition for (i) how an initial aggregate volatility σ_0^s can appear, and (ii) three results in Proposition 1.[11] For that purpose, we simplify the economic environment and make the following assumptions:

A.1 A shock dZ_t at each period takes one of two values: $\{+1, -1\}$ with equal probability.

A.2 Martingale equilibrium: the output gap $\hat{Y}_t$ equals the conditional expected value of the next-period gap $\hat{Y}_{t+1}$. Thus, if $\hat{Y}_{t+1}$ takes either $\hat{Y}_{t+1}^{(1)}$ or $\hat{Y}_{t+1}^{(2)}$ when $dZ_{t+1} = 1$ or -1, respectively, then

$$\hat{Y}_t = \frac{1}{2}\left(\hat{Y}_{t+1}^{(1)} + \hat{Y}_{t+1}^{(2)}\right).$$

A.3 Aggregate demand (i.e., output gap) $\hat{Y}_t$ falls as the conditional variance of the next-period's $\hat{Y}_{t+1}$ rises, due to precautionary savings. $\hat{Y}_t$ and σ_t^s are zero on the stabilized path (i.e., flexible-price economy).

Since we have two possible realizations of shock dZ_t at each period, we can draw a tree diagram as depicted in Figure 1. The thick vertical line represents the stabilized path, with areas at its left and right representing recessions and booms, respectively. The key to build a rational expectations equilibrium supporting a self-fulfilling rise in volatility $\sigma_0^s > 0$ is to construct the agents' path-dependent consumption strategy with time-varying conditional volatilities.

First, let us imagine that the the current period agents (Agents$_0$) suddenly believe that the future agents will choose the path-dependent consumption demand[12] so that the next-period's $\hat{Y}_1$ becomes $\hat{Y}_1^{(1)}$ after $dZ_1 = +1$ is realized and $\hat{Y}_1^{(2)}$ if $dZ_1 = -1$ is realized, with $\hat{Y}_1^{(1)} > \hat{Y}_1^{(2)}$. Then the current output $\hat{Y}_0$ becomes $\hat{Y}_0 = \frac{1}{2}\left(\hat{Y}_1^{(1)} + \hat{Y}_1^{(2)}\right)$ with $\hat{Y}_0$ below the

[11] We provide simulation results from an alternative calibrated model with stock markets in Online Appendix B.5.

[12] Note that agents' demand determines output in this environment with rigid prices.

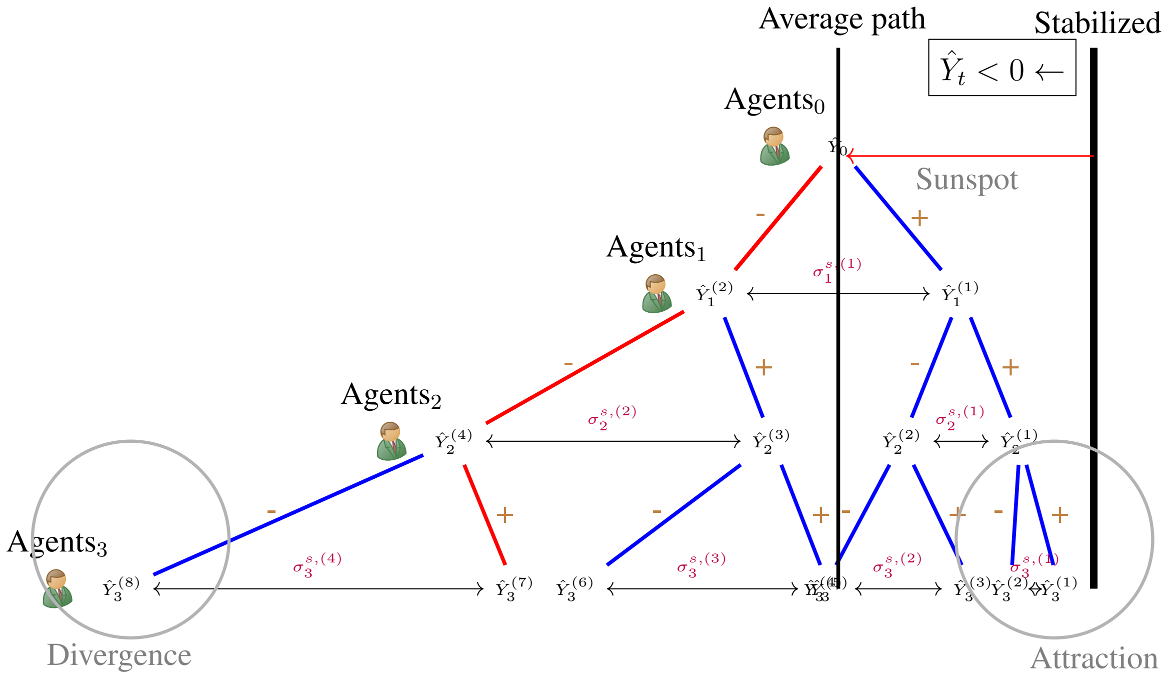

Figure 1: A rise in σ_0^s as a rational expectations equilibrium

stabilized path, as Agents$_0$ believe there exists dispersion in next-period outcomes, which is given as $\sigma_1^{s,(1)} = \frac{\hat{Y}_1^{(1)} - \hat{Y}_1^{(2)}}{2}$, which leads to lower consumption through precautionary savings at $t = 0$. Imagine $dZ_1 = -1$ is realized. For Agents$_0$'s belief that $\hat{Y}_1 = \hat{Y}_1^{(2)}$ to be consistent, Agents$_1$ must believe that future agents will choose their consumption in a way that at time 2, $\hat{Y}_2$ becomes $\hat{Y}_2^{(3)}$ with $dZ_2 = +1$ and $\hat{Y}_2^{(4)}$ with $dZ_2 = -1$, with conditional volatility $\sigma_2^{s,(2)} = \frac{\hat{Y}_2^{(3)} - \hat{Y}_2^{(4)}}{2}$ higher than $\sigma_1^{s,(1)}$, since $\hat{Y}_1^{(2)}$ is lower than the initial output $\hat{Y}_0$.

After dZ_2 is realized, Agents$_1$'s belief about $\hat{Y}_2$ can be made consistent through future agents $\{\text{Agents}_{n \geq 2}\}$'s coordination in a forward looking fashion. Observe that all the nodes in Figure 1 satisfy assumptions A.2 and A.3, with distance between adjacent nodes getting progressively narrower (wider) as output gap gets closer (farther) to the stabilization. This results in divergent and attraction paths balancing each other out, and in expectation, output gap $\{\hat{Y}_t\}$ follows a martingale process. In sum, Agents$_0$'s initial doubt that the next-period's outcome will be volatile is made consistent by coordination between intertemporal agents (i.e., the representative household) at each node.[13]

[13] Our equilibrium construction is feasible since all future agents share the common knowledge of their consumption strategies and there is no friction in communication among agents in intertemporal periods (i.e., perfect recall). For how limited recall removes indeterminacy, see Angeletos and Lian (2023).

Note that (i) we obtain an equilibrium with a *stochastic* aggregate volatility: i.e., σ_t^s is dependent on the path of shocks, as output gap $\{\hat{Y}_t\}$ is stochastic and negatively depends on the conditional volatility of its next-period level. Equation (14) specifies the exact stochastic process of $\{\sigma_t^s\}$ starting from $\sigma_0^s > 0$; (ii) Since volatility σ_t^s decreases as output gap $\hat{Y}_t$ approaches the stabilized path, this path becomes an attraction point for the set of alternative paths in its neighborhood, justifying the result of Proposition 1 that σ_t^s almost surely converges to zero over time. Nonetheless, as volatility σ_t^s rises whenever output $\hat{Y}_t$ deviates farther from the stabilized level, this also aligns with the result of Proposition 1 that maximal σ_t^s diverges, $\mathbb{E}_0(\max_t(\sigma_t^s)^2) = \infty$. However, this divergent behavior only happens with vanishingly small probability as $\sigma_t^s \overset{a.s}{\to} 0$.

The conclusion in terms of monetary policy is that a conventional Taylor rule almost surely stabilizes the disruption caused by an initial volatility shock $\sigma_0^s > 0$ in the long-run, but does not prevent the economy from entering a crisis phase with excess volatility path $\{\sigma_t^s\}$ starting from σ_0^s.

Escape clause If the central bank and/or the government credibly commit to prevent $\hat{Y}_t$ from going below a predetermined threshold through interventions,[14] these equilibria arising from the aggregate volatility σ_0^s supported by paths in Figure 1 (i.e., martingale equilibrium) are not sustained anymore as a possible rational expectations equilibrium (REE). This escape clause illustrates how the credible commitment of government entities to intervene whenever the economy probabilistically enters a big recession actually preclude the possibility of a crisis phase initiated by the positive volatility shock $\sigma_0^s > 0$. Whether this type of commitment from government and central bank is credible is important, as absolute credibility is required to prevent the apparition of an equilibrium with $\sigma_0^s > 0$.

Also, note that with $\phi_y \to \infty$, our martingale equilibrium with a given initial volatility $\sigma_0^s > 0$ uniformly converges to the stabilized path, i.e., $\hat{Y}_t = 0$.

Negative volatility We can similarly construct a rational expectations equilibrium with an initial negative self-fulfilling volatility $\sigma_0^s < 0$. This equilibrium is characterized by a

[14]For example, governments might commit to incur huge fiscal deficits whenever the economy undergoes a severe recession. This prescription entails similar implications about what governments can do to restore determinate equilibrium as in Benhabib et al. (2002), who especially deal with the role of monetary-fiscal regimes in regards to eliminating indeterminacy posed by ZLB. In a similar way, Obstfeld and Rogoff (2021) show how a probabilistic (and small) fiscal currency backing by the government rules out speculative hyperinflation in monetary models.

boom with strong aggregate demand and low volatility.[15] Therefore, we conclude that our non-linear characterization of the model generates the reasonable prediction of (i) appearance of boom/crisis phases coming from self-fulfilling volatility shocks, and (ii) the joint evolution of the first (output level) and second (conditional volatility) order moments of the model during crises and booms.

2.2 A New Monetary Policy

Let's assume, instead, that the central bank follows this alternative policy rule:

$$i_t = r^n + \phi_y \hat{Y}_t - \underbrace{\frac{1}{2}\left((\sigma+\sigma_t^s)^2 - \sigma^2\right)}_{\text{Aggregate volatility targeting}}, \quad \text{where} \quad \phi_y > 0, \tag{15}$$

which, in addition to output gap $\hat{Y}_t$, targets the aggregate volatility of the output gap with a coefficient $\frac{1}{2}$. By plugging the above policy rule (15) into the IS equation (8), the volatility feedback terms in the drift part cancel out and therefore, we obtain dynamics represented by (12), which guarantees model determinacy and ensures $\hat{Y}_t = 0$, $\forall t$ as the unique rational expectations equilibrium when the Taylor principle $\phi_y > 0$ is satisfied. Therefore, we conclude that monetary policy following (15) eliminates the potential for the appearance of self-fulfilling aggregate volatility.

Interpretation The additional volatility target in the policy rule is necessary to offset the feedback channel between the endogenous volatility of the output gap and its drift. To get a more intuitive interpretation of this result, we can rearrange equation (15) as $i_t = r_t^T + \phi_y \hat{Y}_t$ where r_t^T is the risk-adjusted natural rate defined in equation (9). Therefore, an alternative interpretation is that monetary policy in a risky environment should target the risk-adjusted, and not simply the natural, interest rate. Note that r_t^T in our environment is time-varying, as it depends on the potential excess volatility σ_t^s.

[15] As seen in equation (6), the actual output Y_t's process features $\sigma + \sigma_t^s$ as its conditional volatility. Thus, a self-created negative excess volatility $\sigma_0^s < 0$ reduces the volatility of the growth rate of Y_t from σ to $\sigma + \sigma_t^s$.

Reformulation We can rewrite our new policy rule in (15) as

$$\underbrace{\not\rho}_{\text{Discount rate}} + \underbrace{\frac{\mathbb{E}_t\left(d\log Y_t\right)}{dt}}_{\text{Growth rate}} = \underbrace{\not\rho}_{\text{Discount rate}} + \underbrace{\frac{\mathbb{E}_t\left(d\log Y_t^n\right)}{dt}}_{\substack{\text{Benchmark}\\ \text{growth rate}}} + \underbrace{\phi_y \hat{Y}_t}_{\text{Business cycle targeting}} . \tag{16}$$

Our modified policy rule targeting the aggregate volatility σ_t^s as prescribed in equation (15) thus can be interpreted as a rule on the rate of change of $\log$-consumption (or output) as a function of the traditional output gap target. Basically, the rate that determines the households' intertemporal substitution should account for the precautionary behavior stemming from aggregate volatility, instead of just the risk-free policy rate i_t, and therefore in order to achieve determinacy as well as stabilization in our model, the expected growth rate of consumption (or output) must target business cycle fluctuations.

Practicality A potential issue with the policy rule in (15) is its lack of robustness to practical implementation, as it necessitates extremely precise targeting of the aggregate volatility component $(\sigma + \sigma_t^s)^2 - \sigma^2$. Failing this precision, the rule cannot counteract the precautionary savings feedback loop present in the non-linear IS equation (8), which sustains and propagates the initial volatility shock σ_0^s. In reality, the components of output volatility $\{\sigma,\, \sigma_t^s\}$ and the risk-adjusted natural rate r_t^T may not be directly observable (or observable without error).[16] Similarly, the coefficient preceding the volatility term, indicative of the policymakers' response strength, must be precisely $\frac{1}{2}$. To understand the consequences of deviating from the $\frac{1}{2}$ volatility target, we consider the following alternative rule:

$$i_t = r^n + \phi_y \hat{Y}_t - \phi_{\text{vol}}\left((\sigma + \sigma_t^s)^2 - \sigma^2\right), \tag{17}$$

where ϕ_{vol} is a constant term, potentially different from $\frac{1}{2}$. With the policy rule in (17), we obtain

$$d\hat{Y}_t = \left(\phi_y \hat{Y}_t + \left(\frac{1}{2} - \phi_{\text{vol}}\right)\left((\sigma + \sigma_t^s)^2 - \sigma^2\right)\right) dt + \sigma_t^s dZ_t. \tag{18}$$

[16] As an illustration, assume a multiplicative measurement error for the volatility gap $\equiv (\sigma + \sigma_t^s)^2 - \sigma^2$ such that volatility $\text{gap}_t^{\text{obs}} = \varepsilon_t \cdot$ volatility gap_t, where volatility $\text{gap}_t^{\text{obs}}$ represents the measured volatility gap. In those cases, even with the precise targeting strength of $\frac{1}{2}$ on the observed volatility gap, i.e., volatility $\text{gap}_t^{\text{obs}}$, central banks effectively deviate from $\frac{1}{2}$ response strength on the true volatility gap.

as the new $\{\hat{Y}_t\}$ dynamics. With $\phi_{\text{vol}} \neq \frac{1}{2}$, the martingale equilibrium with self-fulfilling volatility σ_t^s reappears and is characterized by[17]

$$\hat{Y}_t = -\frac{(\sigma + \sigma_t^s)^2}{2\phi_{\text{total}}} + \frac{\sigma^2}{2\phi_{\text{total}}}, \quad \text{with} \quad \phi_{\text{total}} \equiv \frac{\phi_y}{1 - 2\phi_{\text{vol}}}, \tag{19}$$

where $\{\sigma_t^s\}$'s process after an initial volatility shock σ_0^s appears is given by

$$d\sigma_t^s = -\frac{\phi_{\text{total}}^2 (\sigma_t^s)^2}{2(\sigma + \sigma_t^s)^3} dt - \phi_{\text{total}} \frac{\sigma_t^s}{\sigma + \sigma_t^s} dZ_t. \tag{20}$$

Note that $\phi_{\text{vol}} \to \frac{1}{2}$, given $\phi_y > 0$, is equivalent to $\phi_y \to \infty$ with $\phi_{\text{vol}} = 0$, both of which lead to $\phi_{\text{total}} \to \infty$ and guarantee determinacy. Therefore, there exists an alternative -albeit equally impractical- stabilization rule that involves an infinitely aggressive off-equilibrium threat to output gap deviations.[18] In Online Appendix B, we analyze how the relative size of coefficients ϕ_y and ϕ_{vol} affects the pace of stabilization after a self-fulfilling volatility shock σ_0^s appears.

Comparison Woodford (2001, 2003) study the Taylor rule in a log-linearized New Keynesian model summarized by[19,20]

$$\begin{aligned} \mathbb{E}_t(d\hat{Y}_{t+1}) &= (i_t^m - r^n)\, dt, \\ i_t &= i_t^* + \phi_y \hat{Y}_t, \end{aligned} \tag{21}$$

where i_t^m is the interest rate that governs the household's intertemporal consumption smoothing, and i_t^* the central target of the policy rate i_t. They uncover that:

B.1 When i_t^m is equal to i_t, then $i_t^* = r^n$ guarantees $\hat{Y}_t = 0$ as unique equilibrium. Even if $i_t^* \neq r^n$, we still have a unique equilibrium, but $\hat{Y}_t \neq 0$ on the equilibrium path.

B.2 When $i_t^m \neq i_t$, $i_t^* = r^n + (i_t - i_t^m)$ achieves $\hat{Y}_t = 0$ as unique equilibrium. If $i_t - i_t^m$ is an exogenous process, then even when $i_t^* \neq r^n + (i_t - i_t^m)$, we still have a unique equilibrium, but $\hat{Y}_t \neq 0$ on the equilibrium path.

[17] Equations (19) and (20) are easily derived in a similar way to Proposition 1.

[18] See Cochrane (2007) for a comprehensive discussion on this topic in traditional New-Keynesian frameworks.

[19] For comparison, inflation is abstracted away in (21).

[20] We thank an anonymous referee for suggesting this comparison.

What we do corresponds to neither case: in our model, $i_t - i_t^m$ depends on the endogenous volatility of the $\{\hat{Y}_t\}$ process, with $r_t^T \equiv r^n + (i_t - i_t^m)$ in equation (9). We show that

C.1 If $i_t^* = r_t^T$, we achieve $\hat{Y}_t = 0$ as a unique equilibrium. In this case, the policy rule corresponds to the new rule proposed in (15).

C.2 In contrast to Woodford (2001, 2003) where $i_t - i_t^m$ is exogenous, now if $i_t^* \neq r_t^T$, we cannot guarantee a unique equilibrium, and the martingale equilibrium of Section 2.1 with self-fulfilling initial volatility σ_0^s potentially appears.

C.3 $i_t - i_t^m$ depends only on the volatility gap, i.e., $(\sigma + \sigma_t^s)^2 - \sigma^2$. Thus in a knife-edge case where $i_t^* - (i_t - i_t^m)$ does not contain any multiple of the volatility gap (or more generally, is not a function of the (excess) volatility σ_t^s), even if $i_t^* - (i_t - i_t^m) \neq r^n$, we have a unique equilibrium, but $\hat{Y}_t \neq 0$ along the equilibrium path.

Model with Stock Markets In Online Appendix B, we provide a model that incorporates stock markets and inflation, and show the commonly observed measures of *financial volatility* or *risk-premium* can serve as a proxy for the implementation of our new policy rule in (15).

The model showcases a similar role for aggregate stock price volatility and risk premium in business cycle fluctuations: a more volatile stock market with a higher risk premium reduces aggregate financial wealth through individual investors' portfolio decisions, diminishing aggregate demand and output in the presence of nominal rigidities. Due to the analogous mathematical structure concerning the influence of aggregate volatility on aggregate demand, we can construct an equilibrium in which aggregate stock price volatility is generated in a self-fulfilling manner under conventional Taylor rules. Then we argue that a *generalized* Taylor rule that has the same form as (15) and targets risk-premium achieves what we call *ultra-divine coincidence*: the simultaneous stabilization of inflation, output gap, and risk-premium (equivalently, aggregate stock price volatility). Furthermore, the impulse responses simulated from our calibrated model, as detailed in Online Appendix B, quantitatively align with the empirical responses derived from a structural Vector Autoregression analysis presented in Online Appendix C.

3 Conclusion

Conventional Taylor rules, even with the aggressive targeting of traditional macroeconomic indicators, cannot guarantee equilibrium determinacy, allowing self-fulfilling aggregate volatility to surface and influence the business cycle. This failure of conventional rules to ensure determinacy arises from their inability to properly target the *expected growth rate of output*, which is pivotal for households' precautionary behavior in making their intertemporal substitution decisions. We propose an alternative monetary rule that restores determinacy by targeting not only the conventional mandates, such as the output gap, but also the volatility of the economy in a specific manner, thus effectively managing the expected growth rate of aggregate output. This new policy rule facilitates the joint stabilization of the output gap and aggregate volatility.

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I Proofs and Derivations

Derivation of equation (3) From the definition of (nominal) state-price density $\xi_t^N = e^{-\rho t}\frac{1}{C_t}\frac{1}{p_t}$, we obtain

$$\frac{d\xi_t^N}{\xi_t^N} = -\rho dt - \frac{dC_t}{C_t} - \frac{dp_t}{p_t} + \left(\frac{dC_t}{C_t}\right)^2 + \left(\frac{dp_t}{p_t}\right)^2 + \frac{dC_t}{C_t}\frac{dp_t}{p_t}. \tag{I.1}$$

Since we have a perfectly rigid price (i.e., $p_t = \bar{p}$ for $\forall t$), the above (I.1) becomes

$$\frac{d\xi_t^N}{\xi_t^N} = -\rho dt - \frac{dC_t}{C_t} + \left(\frac{dC_t}{C_t}\right)^2 \tag{I.2}$$

$$= -\rho dt - \frac{dC_t}{C_t} + \mathrm{Var}_t\left(\frac{dC_t}{C_t}\right). \tag{I.3}$$

Plugging equation (I.2) into equation (2), we obtain

$$\mathbb{E}_t\left(\frac{dC_t}{C_t}\right) = (i_t - \rho)\, dt + \mathrm{Var}_t\left(\frac{dC_t}{C_t}\right). \tag{I.4}$$

Derivation of equation (8) From equation (7), we obtain

$$d\ln Y_t = \left(i_t - \rho + \frac{1}{2}\left(\sigma + \sigma_t^s\right)^2\right) dt + (\sigma + \sigma_t^s) dZ_t. \tag{I.5}$$

From (5), we obtain

$$d\ln Y_t^n = \left(r^n - \rho + \frac{1}{2}\sigma^2\right) dt + \sigma dZ_t. \tag{I.6}$$

Therefore, by subtracting equation (I.6) from equation (I.5), we obtain

$$d\hat{Y}_t = \left(i_t - \left(r^n - \frac{1}{2}\left(\sigma + \sigma_t^s\right)^2 + \frac{1}{2}\sigma^2\right)\right) dt + \sigma_t^s dZ_t, \tag{I.7}$$

which derives equation (8).

Proof of Proposition 1. From equation (14), $\{\sigma_t^s\}$ process can be written as

$$d\sigma_t^s = -(\phi_y)^2 \frac{(\sigma_t^s)^2}{2(\sigma + \sigma_t^s)^3} dt - \phi_y \frac{\sigma_t^s}{\sigma + \sigma_t^s} dZ_t. \tag{I.8}$$

Using Ito's lemma, we get the process for $(\sigma + \sigma_t^s)^2$ which is a martingale, as given by

$$
\begin{aligned}
d(\sigma+\sigma_t^s)^2 &= 2(\sigma+\sigma_t^s)d\sigma_t^s + (d\sigma_t^s)^2 \\
&= 2(\sigma+\sigma_t^s)\left(-\frac{(\phi_y)^2(\sigma_t^s)^2}{2(\sigma+\sigma_t^s)^3}dt - \phi_y\frac{\sigma_t^s}{\sigma+\sigma_t^s}dZ_t\right) + (\phi_y)^2\frac{(\sigma_t^s)^2}{(\sigma+\sigma_t^s)^2}dt \qquad \text{(I.9)} \\
&= -2\phi_y(\sigma_t^s)dZ_t.
\end{aligned}
$$

Therefore, we have $\mathbb{E}_0((\sigma+\sigma_t^s)^2) = (\sigma+\sigma_0^s)^2$. By applying Doob's martingale convergence theorem as $(\sigma+\sigma_t^s)^2 \geq 0, \forall t$, we know $\sigma_t^s \overset{a.s}{\to} \sigma_\infty^s = 0$ since:

$$
\underbrace{d\sigma_t^s}_{\overset{a.s}{\to}0} = -\frac{(\phi_y)^2(\sigma_t^s)^2}{\underbrace{2(\sigma+\sigma_t^s)^3}_{\overset{a.s}{\to}0}}dt - \phi_y\frac{\sigma_t^s}{\underbrace{\sigma+\sigma_t^s}_{\overset{a.s}{\to}0}}dZ_t. \qquad \text{(I.10)}
$$

Thus equation (I.10) proves $\sigma_t^s \overset{a.s}{\to} \sigma_\infty^s = 0$. From equation (13) $\sigma_t^s \overset{a.s}{\to} \sigma_\infty^q = 0$ leads to $\hat{Y}_t \overset{a.s}{\to} 0$. Finally, we have $\mathbb{E}_0(\max_t(\sigma_t^s)^2) = \infty$, since otherwise the uniform integrability says $\mathbb{E}_0((\sigma+\sigma_\infty^s)^2) = (\sigma+\sigma_0^s)^2$, which is a contradiction to our earlier result $\sigma_t^s \overset{a.s}{\to} 0$ since $\sigma_\infty^s = 0$ and $\sigma_0^s > 0$ by assumption in Proposition 1.

■

A Detailed Derivations in Section 2

A.1 Model Setup

A representative household solves the following intertemporal optimization consumption-savings decision problem:

$$\max_{\{C_s,L_s\}_{s\geq t}} \mathbb{E}_t \int_s^{\infty} e^{-\rho(s-t)} \left[\log C_s - \frac{L_s^{1+\frac{1}{\eta}}}{1+\frac{1}{\eta}}\right] \mathrm{d}s \quad \text{s.t.} \quad \mathrm{d}B_t = \left[i_t B_t - p_t C_t + w_t L_t + D_t\right] \mathrm{d}t$$

where C_t is consumption, L_t aggregate labor, w_t is the equilibrium wage level, B_t are risk-free bonds held by the household at the beginning of t (hence, B_t at t is taken as given for each household), i_t is the nominal interest rate, D_t is a lump-sum transfer of any firm profits/losses towards the household, p_t the nominal price of consumption goods and ρ is the subjective discount rate of the household.

An individual firm i produces in this economy with the following production function:

$$Y_t^i = A_t L_t^i, \quad \text{where}$$
$$\frac{\mathrm{d}A_t}{A_t} = g\mathrm{d}t + \underbrace{\sigma}_{\text{Fundamental risk}} \mathrm{d}Z_t$$

where A_t is the economy's total factor productivity, assumed to be exogenous and to follow a geometric Brownian motion with drift, where g is the expected growth rate of A_t, σ is its volatility, which we assume to be constant over time and call *fundamental* volatility, and Z_t is a standard Brownian motion process. It follows that firms' profits are defined as:

$$D_t = p_t Y_t - w_t L_t$$

Finally, we assume that in equilibrium, bonds are in zero net supply (i.e. $B_t = 0, \ \forall t$) and that there is no government spending, so market clearing in this economy results in $C_t = Y_t$.

A.2 Flexible Price Economy

We first solve the flexible price economy as our benchmark economy. In that purpose, we assume the usual Dixit Stiglitz monopolistic competition among firms, where the demand each firm i faces is given by

$$D(p_t^i, p_t) = \left(\frac{p_t^i}{p_t}\right)^{-\varepsilon} Y_t,$$

where p_t^i is an individual firm i's price, p_t is the price aggregator, and Y_t is the aggregate output. Each firm i takes the aggregate price p_t, wage w_t, and the aggregate output Y_t as given.

A.2.1 Household problem

In the flexible price economy, each household takes $\{A_t, p_t, i_t\}$ process as given:

$$\frac{dp_t}{p_t} = \pi_t dt + \sigma_t^p dZ_t \tag{A.1}$$

and

$$di_t = \mu_t^i dt + \sigma_t^i dZ_t \tag{A.2}$$

where π_t, σ_t^p, μ_t^i, and σ_t^i are all endogenous, so the state variable for each household would become $\{B_t, A_t, p_t, i_t\}$.[1]

Hamilton-Jacobi-Bellman (HJB) formulation of the households' problem We define the value function as:

$$\Gamma \equiv \Gamma\left(B_t, A_t, p_t, i_t, t\right) = \max_{\{C_s, L_s\}_{s\geq t}} \mathbb{E}_t \int_s^\infty e^{-\rho(s-t)} \left[\log C_s - \frac{L_s^{1+\frac{1}{\eta}}}{1+\frac{1}{\eta}}\right] \mathrm{d}s.$$

The formula for the stochastic HJB equation is given as:

$$\rho \cdot \Gamma = \max_{C_t, L_t} \left\{ \log C_t - \frac{L_t^{1+\frac{1}{\eta}}}{1+\frac{1}{\eta}} + \frac{\mathbb{E}_t\left[\mathrm{d}\Gamma\right]}{\mathrm{d}t} \right\}. \tag{A.3}$$

Using Ito's Lemma, we compute:

$$d\Gamma = \mu_t^\Gamma dt + \sigma_t^\Gamma dZ_t \tag{A.4}$$

[1] This is a conjectural but correct statement due to the classical dichotomy between real and nominal sectors: output, consumption, and labor in equilibrium turn out to depend on A_t only and it turns out that p_t and i_t do not matter for the real economy and the welfare of the households.

where

$$\begin{aligned}\mu_t^\Gamma =&\Gamma_t + \Gamma_B \cdot (i_t B_t - p_t C_t + w_t L_t + D_t) + \Gamma_A \cdot A_t g + \Gamma_p \cdot p_t \pi_t + \Gamma_i \cdot \mu_t^i \\ &+ \frac{1}{2}\Gamma_{AA} \cdot (A_t \sigma)^2 + \frac{1}{2}\Gamma_{pp} \cdot (p_t \sigma_t^p)^2 + \frac{1}{2}\Gamma_{ii} \cdot (\sigma_t^i)^2 \\ &+ \Gamma_{Ap} \cdot (\sigma A_t)(p_t \sigma_t^p) + \Gamma_{Ai} \cdot (\sigma A_t)\sigma_t^i + \Gamma_{pi} \cdot (p_t \sigma_t^p)\sigma_t^i\end{aligned} \tag{A.5}$$

and $\sigma_t^\Gamma = \Gamma_A(\sigma A_t) + \Gamma_p(p_t \sigma_t^p) + \Gamma_i(\sigma_t^i)$. In the same way, we compute $d\Gamma_B = \mu_t^{\Gamma_B} dt + \sigma_t^{\Gamma_B} dZ_t$ where

$$\begin{aligned}\mu_t^{\Gamma_B} =&\Gamma_{Bt} + \Gamma_{BB} \cdot (i_t B_t - p_t C_t + w_t L_t + D_t) + \Gamma_{BA} \cdot A_t g + \Gamma_{Bp} \cdot p_t \pi_t + \Gamma_{Bi} \cdot \mu_t^i \\ &+ \frac{1}{2}\Gamma_{BAA} \cdot (A_t \sigma)^2 + \frac{1}{2}\Gamma_{Bpp} \cdot (p_t \sigma_t^p)^2 + \frac{1}{2}\Gamma_{Bii} \cdot (\sigma_t^i)^2 \\ &+ \Gamma_{BAp} \cdot (\sigma A_t)(p_t \sigma_t^p) + \Gamma_{BAi} \cdot (\sigma A_t)\sigma_t^i + \Gamma_{Bpi} \cdot (p_t \sigma_t^p)\sigma_t^i\end{aligned} \tag{A.6}$$

and $\sigma_t^{\Gamma_B} = \Gamma_{BA}(\sigma A_t) + \Gamma_{Bp}(p_t \sigma_t^p) + \Gamma_{Bi}(\sigma_t^i)$. Note $\Gamma_\Delta = \frac{\partial \Gamma}{\partial \Delta}$ is defined as the derivative with respect to any subindex variable $\Delta = \{t, B, A, p, i\}$. Now plug equation (A.4) into equation (A.3) to obtain:

$$\rho \cdot \Gamma = \max_{C_t, L_t} \left\{ \log C_t - \frac{L_t^{1+\frac{1}{\eta}}}{1+\frac{1}{\eta}} + \mu_t^\Gamma \right\}. \tag{A.7}$$

Households' first-order conditions (FOC) Computing the first-order conditions with respect to C_t and L_t from equation (A.7), we obtain:

$$\Gamma_B = \frac{1}{p_t C_t} \tag{A.8}$$

$$\Gamma_B = \frac{L_t^{\frac{1}{\eta}}}{w_t} \tag{A.9}$$

Finally, merging (A.8) with (A.9) gives us the optimality condition.

State price density and pricing kernel We know the state price density and the stochastic discount factor between two adjacent periods are given by $\zeta_t^N = e^{-\rho t} \frac{1}{p_t C_t}$, and $dQ_t = \frac{d\zeta_t^N}{\zeta_t^N}$, respectively. Let us use a star superscript to denote the choice variables evaluated at the optimum, that is C_t^* and L_t^*. Then, we can express equation (A.7) as:

$$\rho \cdot \Gamma = \log C_t^* - \frac{(L_t^*)^{1+\frac{1}{\eta}}}{1+\frac{1}{\eta}} + \mu_t^{\Gamma,*} \tag{A.10}$$

Taking the derivative of both sides of equation (A.10) with respect to B_t, using the envelop theorem and rearranging, we obtain:

$$(\rho - i_t) \cdot \Gamma_B = \mu_t^{\Gamma_B,*} \tag{A.11}$$

where $\mu_t^{\Gamma_B,*}$ is from equation (A.6) and it is evaluated at the optimum. Plugging (A.11) into the process for Γ_B, we obtain a simplified expression:

$$\mathrm{d}\Gamma_B = (\rho - i_t) \cdot \Gamma_B \mathrm{d}t + \underbrace{\left(\Gamma_{BA}(A_t\sigma) + \Gamma_{Bp}(p_t\sigma_t^p) + \Gamma_{Bi}\left(\sigma_t^i\right)\right)}_{\equiv \sigma_t^{\Gamma_B}} \mathrm{d}Z_t \tag{A.12}$$

Notice that $\zeta_t^N = e^{-\rho t}\Gamma_B$, then, using equation (A.12) and applying Ito's Lemma, we obtain:

$$\mathrm{d}\zeta_t^N = -\,\zeta_t^N \cdot i_t \mathrm{d}t + \zeta_t^N \cdot \left[\frac{\sigma_t^{\Gamma_B}}{\Gamma_B}\right] \mathrm{d}Z_t$$

From the definition of $\mathrm{d}Q_t$, we obtain:

$$\mathrm{d}Q_t \equiv \frac{\mathrm{d}\zeta_t^N}{\zeta_t^N} = -i_t \mathrm{d}t + \left[\frac{\sigma_t^{\Gamma_B}}{\Gamma_B}\right] \mathrm{d}Z_t \tag{A.13}$$

and $\mathbb{E}_t\left[\mathrm{d}Q_t\right] = -i_t \mathrm{d}t$ follows by taking expectations, which proves (2) in the flexible price equilibrium.

Nominal and real interest rates Prices and consumption would be adapted to the filtration generated by our Brownian motion Z_t process. Let us express the processes for consumption and price as:

$$\mathrm{d}p_t = \pi_t p_t \mathrm{d}t + \sigma_t^p p_t \mathrm{d}Z_t \tag{A.14}$$

$$\mathrm{d}C_t = g_t^C C_t \mathrm{d}t + \sigma_t^C C_t \mathrm{d}Z_t \tag{A.15}$$

where π_t, g_t^C, σ_t^p and σ_t^C are variables to be determined in equilibrium, which can be interpreted as inflation rate, expected consumption growth, and volatilities of prices and consumption processes, respectively. As the real state density is defined as $\zeta_t^r = e^{-\rho t}\frac{1}{C_t}$, the real interest rate r_t is defined by the relation $\mathbb{E}_t\left[\frac{d\zeta_t^r}{\zeta_t^r}\right] = -r_t dt$, similarly to (2).

With (A.15), applying Ito's Lemma to the real state density $\zeta_t^r = e^{-\rho t}\frac{1}{C_t}$ results in

$$\frac{d\zeta_t^r}{\zeta_t^r} = -\underbrace{\left[\rho + g_t^C - \left(\sigma_t^C\right)^2\right]}_{\equiv r_t} \mathrm{d}t - \sigma_t^C \mathrm{d}Z_t. \tag{A.16}$$

which determines the real interest rate $r_t = \rho + g_t^C - (\sigma_t^C)^2$. We also apply Ito's Lemma to $\zeta_t^N = e^{-\rho t}\frac{1}{p_t C_t}$ and use the above processes for p_t and C_t to obtain:

$$dQ_t \equiv \frac{\mathrm{d}\zeta_t^N}{\zeta_t^N} = -\left[\rho + g_t^C + \pi_t - (\sigma_t^p)^2 - \left(\sigma_t^C\right)^2 - \sigma_t^p\sigma_t^C\right]\mathrm{d}t - \left[\sigma_t^p + \sigma_t^C\right]\mathrm{d}Z_t$$

which can be rearranged as:

$$dQ_t \equiv \frac{\mathrm{d}\zeta_t^N}{\zeta_t^N} = -\underbrace{\left[r_t + \pi_t - \sigma_t^p\left(\sigma_t^C + \sigma_t^p\right)\right]}_{=i_t}\mathrm{d}t - \left[\sigma_t^p + \sigma_t^C\right]\mathrm{d}Z_t \tag{A.17}$$

Comparing equation (A.13) and equation (A.17), we obtain

$$\begin{aligned} i_t &= r_t + \pi_t - \sigma_t^p\left(\sigma_t^C + \sigma_t^p\right), \\ \text{where:} \quad r_t &= \rho + g_t^C - \left(\sigma_t^C\right)^2. \end{aligned}$$

A.2.2 Firm problem and equilibrium

Firm optimization As the demand each firm i faces is given by

$$D(p_t^i, p_t) = \left(\frac{p_t^i}{p_t}\right)^{-\varepsilon} Y_t$$

as usual where p_t^i is an individual firm's price, p_t is the price aggregator, and Y_t is the aggregate output, each firm i solves the following problem:

$$\max_{p_t^i} \quad p_t^i\left(\frac{p_t^i}{p_t}\right)^{-\varepsilon} Y_t - \frac{w_t}{A_t}\left(\frac{p_t^i}{p_t}\right)^{-\varepsilon} Y_t, \tag{A.18}$$

which results in the following first-order condition for the firm:[2]

$$p_t = \left(\frac{\varepsilon}{\varepsilon - 1}\right)\frac{w_t}{A_t}, \tag{A.19}$$

which is intuitive as it tells us that in equilibrium, price is equal to the marginal cost of production multiplied by the constant mark-up, due to the constant elasticity of demand $\varepsilon > 1$. Using equation (A.19) and the equilibrium condition $C_t = Y_t = A_t L_t$ in the first-order condition of the household in (A.8) and (A.9), we obtain $L_t^n = \left(\frac{\varepsilon-1}{\varepsilon}\right)^{\frac{\eta}{\eta+1}}$,[3] which is a constant. This implies: in the flexible price equilibrium, we have $C_t^n = Y_t^n = A_t\left(\frac{\varepsilon-1}{\varepsilon}\right)^{\frac{\eta}{\eta+1}}$. It follows that the stochastic process for Y_t^n is the same as that for A_t as follows:

$$\frac{\mathrm{d}Y_t^n}{Y_t^n} = \frac{\mathrm{d}C_t^n}{C_t^n} = g\mathrm{d}t + \sigma\mathrm{d}Z_t. \tag{A.20}$$

(A.20) implies that the growth rate of consumption and its volatility are $g_t^C = g$ and $\sigma_t^C = \sigma$, so the real interest rate in the flexible price economy, i.e., the natural rate of interest, can be expressed as $r_t^n \equiv r^n = \rho + g - \sigma^2$ from (A.16), which finally gives

$$\frac{\mathrm{d}Y_t^n}{Y_t^n} = \Big(\underbrace{r^n}_{\text{Natural rate}} - \rho + \sigma^2\Big)\mathrm{d}t + \sigma\mathrm{d}Z_t$$

that proves equation (5).

A.3 Rigid Price Economy

We then solve our rigid price economy with $p_t = \bar{p}$ for $\forall t$. First, let us say the rigid price economy's consumption volatility, which we call σ_t^C is given as $\sigma_t^C = \sigma + \sigma_t^s$ (i.e. volatility of flexible price equilibrium in (A.20), plus excess volatility of rigid price equilibrium). Therefore, the consumption process can be written as:

$$\mathrm{d}C_t = g_t^C C_t \mathrm{d}t + (\sigma + \sigma_t^s) C_t \mathrm{d}Z_t. \tag{A.21}$$

And let us conjecture that this endogenous 'excess' volatility σ_t^s follows $d\sigma_t^s = \mu_t^\sigma dt + \sigma_t^\sigma dZ_t$, which turns out to be one of state variables in the rigid price economy. With price

[2] In equilibrium $p_t^i = p_t$ as every firm chooses the same price level.

[3] We impose the superscript n (i.e., natural) in variables to denote that those are the equilibrium values in the flexible price economy.

rigidity (i.e., $p_t = \bar{p}$ for $\forall t$), the agent takes $\{A_t, \sigma_t^s\}$ process as given, so the state variable for each household would become $\{B_t, A_t, \sigma_t^s\}$.[4]

Hamilton-Jacobi-Bellman (HJB) formulation of the households' problem We define the value function as:

$$\Gamma \equiv \Gamma\left(B_t, A_t, \sigma_t^s, t\right) = \max_{\{C_s, L_s\}_{s\geq t}} \mathbb{E}_t \int_s^{\infty} e^{-\rho(s-t)} \left[\log C_s - \frac{L_s^{1+\frac{1}{\eta}}}{1+\frac{1}{\eta}}\right] \mathrm{d}s$$

The formula for the stochastic HJB equation is:

$$\rho \cdot \Gamma = \max_{C_t, L_t} \left\{ \log C_t - \frac{L_t^{1+\frac{1}{\eta}}}{1+\frac{1}{\eta}} + \frac{\mathbb{E}_t\left[\mathrm{d}\Gamma\right]}{\mathrm{d}t} \right\} \tag{A.22}$$

Using Ito's Lemma, we compute:

$$d\Gamma = \mu_t^{\Gamma} dt + \sigma_t^{\Gamma} dZ_t \tag{A.23}$$

where

$$\begin{aligned} \mu_t^{\Gamma} =& \Gamma_t + \Gamma_B \cdot (i_t B_t - \bar{p} \cdot C_t + w_t L_t + D_t) + \Gamma_A \cdot A_t g + \Gamma_\sigma \cdot \mu_t^{\sigma} \\ &+ \frac{1}{2}\Gamma_{AA} \cdot (A_t \sigma)^2 + \frac{1}{2}\Gamma_{\sigma\sigma} \cdot (\sigma_t^{\sigma})^2 + \Gamma_{A\sigma} \cdot (A_t\sigma)(\sigma_t^{\sigma}) \end{aligned} \tag{A.24}$$

and $\sigma_t^{\Gamma} = \Gamma_A(\sigma A_t) + \Gamma_\sigma(\sigma_t^{\sigma})$. Applying Ito's Lemma to Γ_B, we compute $d\Gamma_B = \mu_t^{\Gamma_B} dt + \sigma_t^{\Gamma_B} dZ_t$ where

$$\begin{aligned} \mu_t^{\Gamma_B} =& \Gamma_{Bt} + \Gamma_{BB} \cdot (i_t B_t - \bar{p} \cdot C_t + w_t L_t + D_t) + \Gamma_{BA} \cdot A_t g + \Gamma_{B\sigma} \cdot \mu_t^{\sigma} \\ &+ \frac{1}{2}\Gamma_{BAA} \cdot (A_t \sigma)^2 + \frac{1}{2}\Gamma_{B\sigma\sigma} \cdot (\sigma_t^{\sigma})^2 + \Gamma_{BA\sigma} \cdot (A_t\sigma)(\sigma_t^{\sigma}) \end{aligned} \tag{A.25}$$

and $\sigma_t^{\Gamma_B} = \Gamma_{BA} \cdot (\sigma A_t) + \Gamma_{B\sigma} \cdot \sigma_t^{\sigma}$. Note $\Gamma_\Delta = \frac{\partial \Gamma}{\partial \Delta}$ is defined as the derivative with respect to any subindex variable $\Delta = \{t, B, A, \sigma_t^s\}$. Now plug equation (A.23) into equation (A.22)

[4]This is a conjectural (but correct) statement as the actual output (thereby, consumption and other variables including inflation, nominal interest rate (that follows the Taylor rule), etc) would turn out to only depend on A_t and σ_t^s under our equilibrium construction.

to obtain:

$$\rho \cdot \Gamma = \max_{C_t, L_t} \left\{ \log C_t - \frac{L_t^{1+\frac{1}{\eta}}}{1+\frac{1}{\eta}} + \mu_t^{\Gamma} \right\} \tag{A.26}$$

Households' first-order conditions (FOC) Computing the first-order conditions with respect to C_t and L_t from equation (A.26), we obtain:

$$\Gamma_B = \frac{1}{\bar{p}C_t} \tag{A.27}$$

$$\Gamma_B = \frac{L_t^{\frac{1}{\eta}}}{w_t} \tag{A.28}$$

Finally, merging (A.27) with (A.28) gives us the optimality condition.

State price density and pricing kernel We know the state price density and the stochastic discount factor between two adjacent periods are given by $\zeta_t^N = e^{-\rho t}\frac{1}{\bar{p}C_t}$, and $dQ_t = \frac{\mathrm{d}\zeta_t^N}{\zeta_t^N}$, respectively. Let us use a star superscript to denote the choice variables evaluated at the optimum, that is C_t^* and L_t^*. Then, we can express equation (A.26) as:

$$\rho \cdot \Gamma = \log C_t^* - \frac{(L_t^*)^{1+\frac{1}{\eta}}}{1+\frac{1}{\eta}} + \mu_t^{\Gamma,*} \tag{A.29}$$

Taking the derivative of both sides of equation (A.29) with respect to B_t, using the envelop theorem and rearranging, we obtain:

$$(\rho - i_t) \cdot \Gamma_B = \mu_t^{\Gamma_B,*} \tag{A.30}$$

where $\mu_t^{\Gamma_B,*}$ is from equation (A.25) and it is evaluated at the optimum. Plugging equation (A.30) into the process for Γ_B, we obtain a simplified expression at the optimum:

$$\mathrm{d}\Gamma_B = (\rho - i_t) \cdot \Gamma_B \mathrm{d}t + \underbrace{\left(\Gamma_{BA} \cdot (A_t \sigma) + \Gamma_{B\sigma} \cdot (\sigma_t^{\sigma})\right)}_{\equiv \sigma_t^{\Gamma_B}} \mathrm{d}Z_t \tag{A.31}$$

Notice that $\zeta_t^N = e^{-\rho t}\Gamma_B$, then using equation (A.31) and applying Ito's Lemma, we obtain:

$$\mathrm{d}\zeta_t^N = -\,\zeta_t^N \cdot i_t \mathrm{d}t + \zeta_t^N \cdot \left[\frac{\sigma_t^{\Gamma_B}}{\Gamma_B}\right] \mathrm{d}Z_t$$

From the previous equation, we obtain:

$$dQ_t \equiv \frac{\mathrm{d}\zeta_t^N}{\zeta_t^N} = -i_t \mathrm{d}t + \left[\frac{\sigma_t^{\Gamma_B}}{\Gamma_B}\right] \mathrm{d}Z_t \tag{A.32}$$

and $\mathbb{E}_t\left[\mathrm{d}Q_t\right] = -i_t \mathrm{d}t$ also follows in the rigid price economy by taking conditional expectations.

Verification of the Martingale Equilibrium Now let us verify that our martingale equilibrium, characterized by equations (13) and (14), satisfies our equilibrium conditions derived above. From (13) and (14),

$$\hat{Y}_t = -\frac{(\sigma + \sigma_t^s)^2}{2\phi_y} + \frac{\sigma^2}{2\phi_y}, \tag{A.33}$$

$$\mathrm{d}\sigma_t^s = \underbrace{-(\phi_y)^2 \frac{(\sigma_t^s)^2}{2(\sigma_t + \sigma_t^s)^3}}_{=\mu_t^\sigma} \mathrm{d}t \underbrace{-\phi_y \left(\frac{\sigma_t^s}{\sigma_t + \sigma_t^s}\right)}_{=\sigma_t^\sigma} \mathrm{d}Z_t. \tag{A.34}$$

These equations will be a solution to the model, as long as there is no contradiction with the equilibrium conditions. In order to check if (A.33) and (A.34) satisfy the equilibrium conditions, first, the output gap is defined as:

$$\hat{Y}_t = \log\left(\frac{Y_t}{Y_t^n}\right) = \log\left(\frac{C_t}{C_t^n}\right) = \log\left(\frac{C_t}{A_t}\right) - \frac{\eta}{\eta+1}\log\left(\frac{\varepsilon-1}{\varepsilon}\right) \tag{A.35}$$

where the last equality follows from $C_t^n = A_t\left(\frac{\varepsilon-1}{\varepsilon}\right)^{\frac{\eta}{\eta+1}}$, as shown above for the flexible price equilibrium. Combining (A.33) and (A.35), we obtain:

$$C_t = A_t\left(\frac{\varepsilon-1}{\varepsilon}\right)^{\frac{\eta}{\eta+1}} \cdot \exp\left\{-\frac{(\sigma+\sigma_t^s)^2}{2\phi_y} + \frac{\sigma^2}{2\phi_y}\right\}, \tag{A.36}$$

which is a function of A_t and σ_t^s. Under fully sticky prices (i.e. $p_t = \bar{p}$, for $\forall t$), From equation (A.27) we knows

$$\Gamma_B = \frac{1}{\bar{p}C_t}. \tag{A.37}$$

We can now compute the derivative of equation (A.37) with respect to A_t and σ_t^s as:

$$\Gamma_{BA} = -\frac{\Gamma_B}{C_t} \cdot \frac{\partial C_t}{\partial A_t}, \tag{A.38}$$

$$\Gamma_{B\sigma} = -\frac{\Gamma_B}{C_t} \cdot \frac{\partial C_t}{\partial \sigma_t^s}. \tag{A.39}$$

Plugging equations (A.38) and (A.39) into equation (A.31), we obtain:

$$\mathrm{d}\Gamma_B = (\rho - i_t) \cdot \Gamma_B \mathrm{d}t - \Gamma_B \left[\frac{A_t}{C_t} \cdot \frac{\partial C_t}{\partial A_t} \cdot \sigma + \frac{1}{C_t} \cdot \frac{\partial C_t}{\partial \sigma_t^s} \cdot \sigma_t^\sigma \right] \mathrm{d}Z_t. \tag{A.40}$$

Using Ito's Lemma in equation (A.37) together with equation (A.21), we obtain

$$\mathrm{d}\Gamma_B = -\Gamma_B \left(g_t^C - (\sigma_t^C)^2 \right) \mathrm{d}t - \Gamma_B(\sigma + \sigma_t^s)\mathrm{d}Z_t. \tag{A.41}$$

Comparing the volatility terms in (A.40) and (A.41) (i.e., terms multiplied to $\mathrm{d}Z_t$), it must follow that:

$$\sigma + \sigma_t^s = \frac{A_t}{C_t} \cdot \frac{\partial C_t}{\partial A_t} \cdot \sigma + \frac{1}{C_t} \cdot \frac{\partial C_t}{\partial \sigma_t^s} \cdot \sigma_t^\sigma. \tag{A.42}$$

We can now compute the derivative of C_t with respect to A_t and σ_t^s as:

$$\frac{\partial C_t}{\partial A_t} = \frac{C_t}{A_t}, \tag{A.43}$$

and

$$\frac{\partial C_t}{\partial \sigma_t^s} = C_t \cdot \left(\frac{-(\sigma + \sigma_t^s)}{\phi_y} \right) = C_t \cdot (\sigma_t^\sigma)^{-1} \cdot \sigma_t^s, \tag{A.44}$$

which satisfies (A.42). Therefore, our martingale equilibrium is verified as an equilibrium.

B The Model with Stock Markets

We now consider a different theoretical framework with explicit stock markets: Two-Agent New-Keynesian model (TANK) based on Dordal i Carreras and Lee (2024), which enables us to analyze the higher-order moments tied to the *aggregate financial volatility*, and provides us the practical implications about monetary policy rules.[5]

B.1 Setting

Time is continuous, and a *filtered* probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\in\mathbb{R}}, \mathbb{P})$ is given as in Section 2. The economy consists of a measure one of capitalists, regarded as neoclassical agents, and the same measure of hand-to-mouth workers, regarded as Keynesian agents. All of the financial wealth is concentrated in the hands of capitalists, while hand-to-mouth workers finance their consumption out of labor income in a similar way to Greenwald et al. (2014).[6] There is a single source of exogenous variation in the aggregate production technology A_t, which is adapted to the filtration $(\mathcal{F}_t)_{t\in\mathbb{R}}$ and evolves according to a geometric process with volatility σ_t:

$$\frac{dA_t}{A_t} = \underbrace{g}_{\text{Growth}}\, dt + \underbrace{\sigma}_{\text{Fundamental risk}}\, dZ_t.$$

We regard the aggregate TFP's volatility σ as the economy's *fundamental* risk, which we take as exogenous. We assume both g and σ to be constant.[7]

B.1.1 Firms and Workers

There are a measure one of monopolistically competitive firms, each producing a differentiated intermediate good $y_t(i)$, $i \in [0,\, 1]$. The final good y_t is constructed by Dixit-Stiglitz aggregator with an elasticity of substitution $\epsilon > 0$ as in

$$y_t = \left(\int_0^1 y_t(i)^{\frac{\epsilon-1}{\epsilon}} di\right)^{\frac{\epsilon}{\epsilon-1}}.$$

[5] All the detailed derivations and proofs are provided in Online Appendix E.

[6] Greenwald et al. (2014) focus on redistributive shock that shift the share of income between labor and capital as a systemic risk for cross-sectional asset pricing. We instead introduce price nominal rigidities in the framework and analyze monetary policy implications.

[7] This assumption is made for simplicity and our analysis can be extended to include cases where σ_t is time-varying and adapted to the Brownian motion Z_t.

An intermediate firm i has the same production function $y_t(i) = A_t(N_{W,t})^{\alpha} n_t(i)^{1-\alpha}$, where $N_{W,t}$ is the economy's aggregate labor, and $n_t(i)$ is the labor demand of an individual firm i at time t. The reason that we introduce a production externality à la Baxter and King (1991) is that it helps us match empirical regularities on asset price and wage co-movements, and it does not affect other qualitative implications of our model.[8] Firm i faces the downward-sloping demand curve $y_i(p_t(i)|p_t, y_t)$, where $p_t(i)$ is the price of its own intermediate good and p_t, y_t are the aggregate price index and output, respectively:

$$y_i(p_t(i)|p_t, y_t) = y_t \left(\frac{p_t(i)}{p_t}\right)^{-\epsilon}.$$

The set of prices charged by intermediate good firms, $\{p_t(i)\}$, is aggregated into the price index p_t as $p_t = \left(\int_0^1 p_t(i)^{1-\epsilon} di\right)^{\frac{1}{1-\epsilon}}$. In contrast to our Section 2 and Appendix A where we assume perfectly rigid prices, we impose a price stickiness à la Calvo (1983), and firms can change prices of their own intermediate goods with δdt probability in a given time interval dt. In the cross-section, this implies that a total δdt portion of firms reset their prices during a given dt time interval.

A representative hand-to-mouth worker supplies labor to intermediate good producers, receives the equilibrium wage income, and spends every dollar he earns on final good consumption. Each worker solves

$$\max_{C_{W,t}, N_{W,t}} \frac{\left(\frac{C_{W,t}}{A_t}\right)^{1-\varphi}}{1-\varphi} - \frac{(N_{W,t})^{1+\chi_0}}{1+\chi_0} \quad \text{s.t.} \quad p_t C_{W,t} = w_t N_{W,t}, \tag{B.1}$$

at every moment t, where $C_{W,t}$, $N_{W,t}$ and w_t are his consumption, labor supply, and equilibrium wage at time t, respectively, and χ_0 is the inverse Frisch elasticity of labor supply. Note that we normalize consumption $C_{W,t}$ by technology A_t, which governs the economy's size.[9] As wage w_t is homogeneous across firms, labor demanded by each firm i, $\{n_t(i)\}$,

[8]In our model, rising asset prices tend to be correlated with the decreasing wage compensation to workers since firm value usually rises if firms can pay less to workers. It violates empirical regularities documented by Chodorow-Reich et al. (2021) in which an increase in stock price tends to push up local aggregate demand variables such as employment and wage. The externality à la Baxter and King (1991) provides us a reasonable calibration that matches these empirical regularities because higher asset prices and aggregate demand raise the firms' marginal product of labor, thus increasing labor demand and wages. Basically, our externality plays similar roles to the capital in the production function.

[9]We introduce the consumption normalization by the aggregate TFP due to the economy's growth. The qualitative results of the model are not affected by this consumption normalization.

are simply combined into aggregate labor $N_{W,t}$ in a linear manner, i.e., $N_{W,t} = \int_0^1 n_t(i)di$.

Final good output y_t can be written as

$$y_t = \frac{A_t N_{W,t}}{\Delta_t}, \quad \text{where} \quad \Delta_t \equiv \left(\int_0^1 \left(\frac{p_t(i)}{p_t} \right)^{-\frac{\epsilon}{1-\alpha}} di \right)^{1-\alpha}. \tag{B.2}$$

where Δ_t is defined as the price dispersion measure. Due to the externality à la Baxter and King (1991), the aggregate production function becomes linear in $N_{W,t}$.

B.1.2 Financial Market and Capitalists

Unlike conventional New-Keynesian models where a representative household owns the firms and receives rebated profits in a lump sum manner, we assume firm profits are capitalized in stock markets as a representative index fund. Capitalists face an optimal portfolio allocation problem involving the allocation of their wealth between the risk-free bond and the stock index at every instant t.

The nominal aggregate financial wealth is $p_t A_t Q_t$, where Q_t is the normalized (or TFP detrended) real asset price. Q_t and p_t are endogenous variables adapted to filtration $(\mathcal{F}_t)_{t\in\mathbb{R}}$ and assumed to evolve according to

$$\frac{dQ_t}{Q_t} = \mu_t^q dt + \underbrace{\sigma_t^q}_{\text{Financial volatility}} dZ_t, \quad \text{and} \quad \frac{dp_t}{p_t} = \pi_t dt + \underbrace{\sigma_t^p}_{\text{Inflation risk}} dZ_t,$$

with endogenous drift μ_t^q and volatility σ_t^q. In particular, we interpret σ_t^q as a measure of financial uncertainty or disruption, as spikes in σ_t^q is empirically observed during a financial crisis. Like Q_t, we assume that the price aggregator p_t follows geometric Brownian motion with drift π_t and volatility σ_t^p. The total financial market wealth $p_t A_t Q_t$ will evolve with a geometric Brownian motion with total volatility $\sigma + \sigma_t^q + \sigma_t^p$. Intuitively, if a capitalist invests in the stock market, they have to bear all three risks: inflation risk, technology risk, and (detrended) real asset price risk.

Note that σ_t^q is determined in equilibrium and can be either positive or negative, i.e., $\sigma_t^q < 0$ corresponds to the case where total real wealth $A_t Q_t$ is less volatile than the TFP process $\{A_t\}$.

In addition to the stock market, we assume that there is a risk-free bond with an associated nominal rate i_t that is controlled by the central bank. Bonds are in zero net supply in equilibrium since all capitalists are equal. A measure one of identical capitalists chooses the portfolio allocation between a risk-free bond and a risky index stock, where in the latter case, they earn the profits of the intermediate goods sector as dividends, as well as the nominal price revaluation of the index due to changes in p_t, A_t and Q_t. Financial markets are competitive, thus each capitalist takes the nominal risk-free rate i_t, the expected stochastic stock market return i_t^m, and the total risk $\sigma + \sigma_t^q + \sigma_t^p$ as given when choosing her portfolio decision.[10] If a capitalist invests a share θ_t of her wealth a_t in the stock market, she bears a total risk $\theta_t a_t(\sigma + \sigma_t^q + \sigma_t^p)$ between t and $t + dt$. Therefore, the riskiness of her portfolio increases proportionally to the investment share θ_t in the index. Capitalists are risk-averse, and ask for a risk-premium compensation $i_t^m - i_t$ when they invest in the risky index market, which is to be determined in equilibrium.

Each capitalist with nominal wealth a_t has $\log$-utility and solves

$$\max_{C_t,\theta_t} \mathbb{E}_0 \int_0^\infty e^{-\rho t} \log C_t dt \;\; \text{s.t.} \;\; da_t = \left(a_t \left(i_t + \theta_t(i_t^m - i_t)\right) - p_t C_t\right) dt + \theta_t a_t \left(\sigma + \sigma_t^q + \sigma_t^p\right) dZ_t, \tag{B.3}$$

where ρ, C_t are her discount rate and final good consumption, respectively. At every instant, she earns returns out of both the risk-free bond and the risky stock investments, and spends on final good consumption.

B.2 Equilibrium and Asset Pricing

Due to the $\log$-utility of capitalists, their nominal state price density ξ_t^N[11] is given by

$$\xi_t^N = e^{-\rho t} \frac{1}{C_t} \frac{1}{p_t}, \;\; \text{where} \;\; \mathbb{E}_t \left(\frac{d\xi_t^N}{\xi_t^N} \right) = -i_t dt \tag{B.4}$$

where the stochastic discount factor between time t (now) and s (future) is by definition given as $\frac{\xi_s^N}{\xi_t^N}$. Aggregate stock market wealth, $p_t A_t Q_t$, is by definition the sum of discounted profit streams from the intermediate goods sector, priced by the above ξ_t^N, as capitalists are marginal stock market investors in equilibrium. We know that at time t, the entire profit of

[10] This competitive market assumption turns out to be an important aspect of our model for explaining inefficiencies caused by aggregate demand externality that individual capitalist's financial investment decision imposes on the aggregate economy. For this issue, see Farhi and Werning (2016).

[11] A superscript N means a nominal state-price density, where a superscript r implies a real one.

the intermediate goods sector is given by

$$D_t \equiv \int (p_t(i)y_t(i) - w_t n_t(i))di = \underbrace{\int p_t(i)y_t(i)di}_{=p_t y_t} - \underbrace{w_t N_{W,t}}_{=p_t C_{W,t}} = p_t(y_t - C_{W,t}) = p_t C_t,$$

where we use the Dixit-Stiglitz aggregator properties that the total expenditure equals a sum of expenditures on intermediate goods and the linear aggregation of labor. Regardless of the price dispersion across firms, the aggregate dividend D_t is equal to the consumption expenditure of capitalists, as workers spend all of their income on consumption.

Plugging the above expressions into the fundamental asset pricing equation yields

$$p_t A_t Q_t = \mathbb{E}_t \frac{1}{\xi_t^N} \int_t^\infty \xi_s^N \left(\underbrace{D_s}_{=p_s C_s \text{ from (B.7)}} \right) ds = \frac{p_t C_t}{\rho}, \tag{B.5}$$

so $p_t C_t = \rho\,(p_t A_t Q_t)$, which is equal to ρa_t in equilibrium with $a_t = p_t A_t Q_t$, i.e., in equilibrium, capitalists hold a wealth amount that equals the total financial market wealth.

Every agent with the same type (i.e., worker or capitalist) is identical and chooses the same decisions in equilibrium. As bonds are in zero net supply, each capitalist's wealth share in the stock market satisfies $\theta_t = 1$, which pins down the equilibrium risk-premium demanded by capitalists. Using (B.3), (B.4), and (B.5), risk-premium is given by

$$\text{rp}_t \equiv i_t^m - i_t = \underbrace{(\sigma + \sigma_t^q + \sigma_t^p)^2}_{\text{Risk-premium}}, \tag{B.6}$$

where rp_t demanded by capitalists increases with either of the three volatilities $\{\sigma_t, \sigma_t^q, \sigma_t^p\}$. As the financial volatility σ_t^q is endogenous, risk-premium rp_t term is endogenous as well and needs to be determined in equilibrium. Note that the wealth gain/loss of the capitalist is equal to the nominal revaluation of the stock. Also note that our equilibrium conditions in (B.5) and (B.6) align with Merton (1971).

We characterize the good's market equilibrium and the equilibrium asset pricing condition of the expected stock return i_t^m as follows: Since capitalists spends ρ portion of their wealth a_t on consumption expenditure and they hold the entire wealth, $C_t = \rho A_t Q_t$ holds in equilibrium. Thus, we can write the equilibrium condition for the final good market as

$$\rho A_t Q_t + \frac{w_t}{p_t} N_{W,t} = \frac{A_t N_{W,t}}{\Delta_t} = y_t. \tag{B.7}$$

The nominal expected return on stock markets i_t^m consists of the dividend yield from the firm profits and the nominal stock price re-valuation (i.e., capital gain) due to fluctuations in $\{p_t, A_t, Q_t\}$. Within our specifications, the dividend yield always is equal to ρ, the discount rate of capitalists. Therefore, when i_t^m changes, only nominal stock prices can adjust endogenously, as the dividend yield is fixed. If we define $\{\mathbf{I}_t^m\}$ as the cumulative stock market return process with $\mathbb{E}_t\left(dI_t^m\right) = i_t^m dt$, the following (B.8) shows the decomposition of i_t^m into dividend yield and stock revaluation in equilibrium:

$$d\mathbf{I}_t^m = \frac{\overbrace{\cancel{p_t}\left(\underbrace{y_t - \frac{w_t}{p_t}N_{W,t}}_{=C_t}\right)}^{\text{Nominal dividend}}}{\underbrace{\cancel{p_t}A_tQ_t}_{\text{Total capital market wealth}}}dt + \underbrace{\frac{d\left(p_tA_tQ_t\right)}{p_tA_tQ_t}}_{\text{Capital gain}} = \rho \cdot dt + \frac{d\left(p_tA_tQ_t\right)}{p_tA_tQ_t} \tag{B.8}$$
$$= \underbrace{\left(\rho + \underbrace{\pi_t}_{\text{Inflation}} + g + \mu_t^q + \sigma_t^q\sigma_t^p + \sigma(\sigma_t^p + \sigma_t^q)\right)}_{=i_t^m} dt + \underbrace{\left(\sigma + \sigma_t^q + \sigma_t^p\right)}_{\text{Risk term}} dZ_t.$$

The equilibrium conditions we have obtained consist of the worker's optimization (i.e., solution of (B.1)), labor aggregation, output formula (i.e., (B.2)), capitalist's optimization (i.e., (B.5) and (B.6)), the good market equilibrium (i.e., (B.7)), and determination of the risky stock return (i.e., (B.8)). To close the model, we also have to derive the supply block of the economy (i.e., pricing decisions of intermediate good firms à la Calvo (1983)) and define the monetary policy rule, which is our most important topic of interest.

The following Lemma B.1 re-derives the Fisher equation when there is a correlation between the (aggregate) price process and the wealth process.

Lemma B.1 (Inflation Premium) *Real interest rate is given by*

$$r_t = i_t - \pi_t + \sigma_t^p \underbrace{\left(\sigma + \sigma_t^p + \sigma_t^q\right)}_{\textit{Wealth volatility}}. \tag{B.9}$$

B.3 Flexible Price Equilibrium

As a benchmark case, we study the flexible price economy. When firms freely reset their prices (i.e., $\delta \to \infty$ case), the real wage $\frac{w_t}{p_t}$ becomes proportional to aggregate technology

A_t. The following proposition summarizes real wage, asset price, natural rate of interest r_t^n (i.e., the real risk-free rate that prevails in the benchmark economy), and consumption process of the capitalist in the flexible price equilibrium. Before we proceed, we define the following parameter, which is the effective labor supply elasticity of workers taking their optimal consumption decision into account.

Definition B.1 *Effective labor supply elasticity of workers:* $\chi^{-1} \equiv \dfrac{1-\varphi}{\chi_0+\varphi}$

Proposition B.1 (Flexible Price Equilibrium) *In the flexible price equilibrium,*[12] *we obtain the analytic characterization of real wage* $\frac{w_t^n}{p_t^n}$*, asset price* Q_t^n*, natural rate of interest* r_t^n*, and consumption of capitalists* C_t^n *as given below:*

1. *The real wage is proportional to aggregate technology* A_t*, and given by*

$$\frac{w_t^n}{p_t^n} = \frac{(\epsilon-1)(1-\alpha)}{\epsilon} A_t$$

2. *The equilibrium detrended asset price* Q_t^n *is constant and given by*

$$Q_t^n = \frac{1}{\rho}\left(\frac{(\epsilon-1)(1-\alpha)}{\epsilon}\right)^{\frac{1}{\chi}}\left(1-\frac{(\epsilon-1)(1-\alpha)}{\epsilon}\right) \quad \text{and} \quad \mu_t^{q,n} = \sigma_t^{q,n} = 0 \tag{B.10}$$

3. *The natural rate* r_t^n *is constant, and given by* $r_t^n \equiv r^n = \rho + g - \sigma^2$*, and consumption of capitalists evolves with*

$$\frac{dC_t^n}{C_t^n} = g dt + \sigma dZ_t = \left(\underbrace{r^n - \rho + \sigma^2}_{\equiv \mu_t^{c,n}}\right) dt + \underbrace{\sigma}_{\equiv \sigma_t^{c.n}} dZ_t, \tag{B.11}$$

In flexible price equilibrium, Proposition B.1 shows that we can characterize closed-form expressions of the real wage $\frac{w_t^n}{p_t^n}$, detrended stock price Q_t^n, and the natural rate r_t^n. First, $\sigma_t^{q,n} = 0$ holds, implying that there is no additional financial volatility running in the economy, in addition to the TFP risk, σ. This feature arises because our economy features no explicit frictions (other than nominal rigidity, which is absent for now) and thus every variable other than the labor supply $N_{W,t}^n$ becomes proportional to A_t. This means that real

[12] We assign superscript n to denote variables in the flexible price (i.e., natural) equilibrium.

wealth $A_t Q_t^n$ has the exactly same volatility as A_t itself, and the financial market imposes no additional risk on the economy. A higher ϵ, the elasticity of substitution, raises the real wage $\frac{w_t^n}{p_t^n}$. It has two competing effects on asset price Q_t^n. A higher real wage reduces the firms' profit as well as the stock price Q_t^n. On the other hand, a higher wage yields a higher labor supply, raising output and stock price Q_t^n. The effective labor supply elasticity χ^{-1} matters in this second effect, thus (B.10) features χ^{-1} exponent on the term that increases with ϵ.

We observe that the natural real interest rate r_t^n is of the same form as (5) in Section 2. Here, a rise in σ raises the stock market's risk-premium level, given by $\text{rp}_t^n \equiv \sigma^2$, as well, inducing capitalists to reduce their portfolio demand for the index, thereby forcing r_t^n to go down in order to prevent a fall in their financial wealth and aggregate demand.

B.4 Sticky Price Equilibrium

When price resetting is sticky à la Calvo (1983), we obtain the Phillips curve that describes inflation dynamics. Since a fixed portion δdt of firms can change their prices on a given infinitesimal time interval dt, we have no stochastic fluctuation in the price process up to a first order, thus $\sigma_t^p = 0$ holds.[13]

A monetary policy rule closes the model. Before analyzing the proper monetary policy rule in this framework, we first describe the 'gap' economy, which is defined as the economy where every variable is in $\log$-deviation from the corresponding level in the flexible price economy. That is, we define any business cycle variable x_t's gap, $\hat{x}_t$, to be the $\log$-deviation of x_t from its natural level x_t^n, which is the level of the variable in the flexible price equilibrium, i.e., $\hat{x}_t \equiv \ln \frac{x_t}{x_t^n}$.

Because the asset price acts as an endogenous aggregate demand shifter, we write every other variable's gap in terms of the asset price gap.[14]

Assumption B.1 (Labor Supply Elasticity) $\chi^{-1} > \frac{\frac{(\epsilon-1)(1-\alpha)}{\epsilon}}{1-\frac{(\epsilon-1)(1-\alpha)}{\epsilon}}$.

Assumption B.1 guarantees the positive co-movement between asset price and other business cycle variables (e.g., real wage and consumptions of capitalists and workers) observed

[13] As in Section 2, we globally characterize the model's demand block, accounting for time-varying higher-order terms. To simplify the analysis, here we linearize the supply block, following Woodford (2003).

[14] Assumption B.1 allows our model to match empirical regularities, and holds under a standard calibration in Table D.1 of Appendix D. Even without Assumption B.1, main features of our model in Appendix B remain unchanged.

in data. With large ϵ, firms' mark-ups decrease and real wage rises as a result. It has a negative impact on the stock price as firm profits decrease, making it harder to satisfy a positive co-movement between gaps in asset price and real wage. A rise in α amplifies the effect of Baxter and King (1991)'s externality, raising labor demand: so that a rise in asset price can result in higher labor demand and real wages. Without Assumption B.1, a positive gap in the asset price depresses wages, labor, and consumption of workers, which can be regarded as a redistributive shock from labor to capital, or in the longer-run, might explain a portion of the observed trend towards increased wealth inequality and income stagnation.[15]

The following Lemma B.2 implies given Assumption B.1, gaps in consumptions of capitalists and workers, asset price, employment, and real wage all co-move with one another up to a first-order. For stabilization purposes, the central bank only needs to deal with the asset price gap $\hat{Q}_t$. From $C_t = \rho A_t Q_t$, we infer that $\hat{Q}_t = \hat{C}_t$. Thus we can interchangeably use $\hat{Q}_t$ or $\hat{C}_t$ to denote gaps of asset price Q_t and consumption of capitalists C_t.

Lemma B.2 (Co-movement) *Given Assumption B.1, gaps in consumption C_t of capitalists, and $C_{W,t}$ of workers, labor supply $N_{W,t}$, and real wage $\frac{w_t}{p_t}$ are positively correlated. Up to a first-order, the following approximation holds:*

$$\hat{Q}_t = \hat{C}_t = \underbrace{\left(\chi^{-1} - \frac{\frac{(\epsilon-1)(1-\alpha)}{\epsilon}}{1 - \frac{(\epsilon-1)(1-\alpha)}{\epsilon}}\right)}_{>0} \frac{\widehat{w_t}}{p_t} = \frac{1}{1+\chi^{-1}} \left(\chi^{-1} - \frac{\frac{(\epsilon-1)(1-\alpha)}{\epsilon}}{1 - \frac{(\epsilon-1)(1-\alpha)}{\epsilon}}\right) \widehat{C_{W,t}}.$$

Using Lemma B.2, we can actually get the following relation between $\hat{Q}_t$ and $\hat{y}_t$.

$$\hat{y}_t = \zeta \hat{Q}_t, \ \ where \ \ \zeta \equiv \chi^{-1} \left(\chi^{-1} - \frac{\frac{(\epsilon-1)(1-\alpha)}{\epsilon}}{1 - \frac{(\epsilon-1)(1-\alpha)}{\epsilon}}\right)^{-1} > 0, \tag{B.12}$$

Proof. Online Appendix E ■

Demand block The dynamic IS equation of $\{\hat{Q}_t\}$ in our model features some important modifications from the canonical New-Keynesian framework. Before we characterize it, we define the risk-premium level $\text{rp}_t \equiv (\sigma + \sigma_t^q)^2$ and its natural level in the flexible price economy $\text{rp}_t^n \equiv \sigma^2$ with $\sigma_t^{q,n} = 0$, as we characterized in equation (B.10). By subtracting rp_t^n from the current risk-premium level rp_t, we define risk-premium gap $\hat{rp}_t \equiv \text{rp}_t - \text{rp}_t^n$.

[15]For instance, Greenwald et al. (2014) interpret redistributive shocks that shift the share of income between labor and capital as a systemic risk to explain various asset pricing phenomena.

Basically, as the risk-premium gap becomes positive in the absence of monetary policy responses, capitalists ask for a higher compensation in bearing financial market risks, causing asset prices to fall below its natural level. We also define the risk-adjusted natural rate r_t^T in the similar way to (9) in Section 2 as $r_t^T \equiv r_t^n - \frac{1}{2}\hat{rp}_t$. r_t^T serves as a real rate anchor for monetary policy. A positive risk-premium gap (i.e., $\hat{rp}_t > 0$), for example, lowers the portfolio demand of capitalists for the stock market compared with the benchmark economy, and thus decreases the anchor rate r_t^T that monetary policy must target for stabilization.

We next characterize the asset price gap $\hat{Q}_t$'s stochastic process. As in equation (8) of Section 2's standard non-linear New-Keynesian framework, the natural rate r_t^n is replaced by the risk-adjusted natural rate r_t^T.

Proposition B.2 (Asset Price Gap Process: IS Equation) *With inflation $\{\pi_t\}$, we obtain*

$$d\hat{Q}_t = (i_t - \pi_t - r_t^T)dt + \sigma_t^q dZ_t, \tag{B.13}$$

where r_t^T takes the role of the natural rate r_t^n. Thus, the aggregate and endogenous financial volatility σ_t^q directly affects the drift of the $\{\hat{Q}_t\}$ process, governing how all other gap variables fluctuate over time.

Proof. Online Appendix E ■

With $\sigma_t^p = 0$, capitalists bear $(\sigma + \sigma_t^q)$ as total risk when investing in the stock market. Due to their $\log$ preference, the risk-premium level rp_t is determined to be $(\sigma + \sigma_t^q)^2$. In the flexible price equilibrium, we have the natural rate given by $r_t^n = r^n = \rho + g - \sigma^2$ and σ_t^q is given by $\sigma_t^{q,n} = 0$. Therefore, the expected real stock market return becomes $r_t^n + \sigma^2 - \frac{1}{2}\sigma^2$, where the factor $\frac{1}{2}\sigma^2$ comes from the quadratic variation factor that arises from the second-order Taylor expansion. In our sticky price equilibrium with endogenous asset price volatility σ_t^q, risk premium changes from σ^2 to $(\sigma + \sigma_t^q)^2$. Thus, with monetary policy rate i_t and inflation π_t, the (real) expected stock market return becomes $i_t - \pi_t + \frac{1}{2}(\sigma + \sigma_t^q)^2$. If this return differs from $r_t^n + \frac{1}{2}\sigma^2$, then $\hat{Q}_t$ endogenously adjusts, and this adjustment creates a real distortion from its effect on aggregate demand.

Equation (B.13) has the same mathematical structure as equation (8) in the standard New-Keynesian model. In Section 2, the endogenous business cycle volatility has a first-order impact on aggregate demand through the household's precautionary savings channel, whereas in the current model with stock markets, the aggregate financial volatility affects risk-premium and financial wealth, determining stock prices and aggregate demand. Due to this isomorphic structure between the two models, we will show that our novel findings in Section 2 continue to hold here, with important implications for monetary policy.

Note that when $\sigma_t^q = \sigma_t^{q,n} = 0$, the risk-adjusted natural rate r_t^T equals the natural rate r_t^n and (B.13) becomes

$$d\hat{C}_t = (i_t - \pi_t - r_t^n)dt, \tag{B.14}$$

which is the IS equation in a standard New-Keynesian model. The crux of the problem is that σ_t^q, which we use as a proxy for financial instability, is itself an endogenous variable to be determined in equilibrium, with no guarantee that it equates its natural level $\sigma_t^{q,n} = 0$.

Supply block We follow the literature on pricing à la Calvo (1983) to determine inflation dynamics. The above Lemma B.2 allows us to express the firms' aggregate marginal cost gap in terms of the asset price gap up to a first order, as asset price determines aggregate demand, which in turn determines such variables as the aggregate marginal cost.

Proposition B.3 (Phillips Curve) *Inflation π_t evolves according to*

$$\mathbb{E}_t d\pi_t = (\rho\pi_t - \frac{\kappa}{\zeta}\hat{y}_t)dt \;\; where \;\; \kappa \equiv \delta(\delta+\rho)\Theta\left(\chi^{-1} - \frac{\frac{(\epsilon-1)(1-\alpha)}{\epsilon}}{1-\frac{(\epsilon-1)(1-\alpha)}{\epsilon}}\right)^{-1}, \quad \Theta = \frac{1-\alpha}{1-\alpha+\alpha\epsilon}. \tag{B.15}$$

Proof. Online Appendix E ■

Plugging equation (B.12) into the Phillips curve, we get $\mathbb{E}_t d\pi_t = (\rho\pi_t - \kappa\hat{Q}_t)dt$, which is expressed in terms of $\hat{Q}_t$. Under Assumption B.1, i.e., $\kappa > 0$, a higher asset price gap $\hat{Q}_t$ means that the economy is over-heated, and thus inflation increases. Note that when the price resetting probability increases (i.e., $\delta \to \infty$), we have $\kappa \to \infty$ and $\hat{Q}_t = 0$ for $\forall t$.

Macroprudential policies There are in general two goals in short (and/or medium)-run macroeconomics: *macro-stabilization* and *financial stability*. Many policymakers believe that financial stability should be dealt with by regulations and macroprudential policies imposed on banks and financial institutions, with business cycle stabilization being the sole focus of monetary policy. Because our model is parsimonious and does not include any complex financial market participants, those macroprudential regulations that tackle potential financial instabilities can be regarded as a policy avenue to prevent σ_t^q from deviating from $\sigma_t^{q,n} = 0$. If $\sigma_t^q = \sigma_t^{q,n} = 0$, then as seen in (B.14), our model features exactly the same dynamics as conventional New-Keynesian models. In that case, a conventional policy rule can solely focus on business cycle stabilization.

One interesting aspect built in our model is that financial stability issues (i.e., volatility and risk-premium) are intertwined with macro-stabilization (i.e., output gap and inflation). Therefore, our question is whether conventional monetary policy rules can achieve financial stability as well as macro stabilization.

B.5 Monetary Policy

We now analyze conventional Taylor rules with inflation and output gap as policy targets. After showing limitations of such policies and how a self-fulfilling financial volatility can arise, we propose a generalized version of the Taylor rule for stochastic environments that successfully achieve the twin objectives of financial stability and macroeconomic stability. Note that the natural rate of interest and the natural risk-premium are given by $r_t^n = r^n = \rho + g - \sigma^2 > 0$ and $\text{rp}_t^n = \text{rp}^n = \sigma^2$.

B.5.1 Old Monetary Rule

Conventional Taylor rule and Bernanke and Gertler (2000) rule We start with a conventional Taylor rule with a constant intercept equal to the natural rate r^n, and the standard inflation and output gap targets, given by $i_t = r^n + \phi_\pi \pi_t + \phi_y \hat{y}_t$, where $\hat{y}_t$ and π_t are output gap and inflation, respectively. Note that we assume a zero trend inflation target. We can rewrite it as

$$i_t = r^n + \phi_\pi \pi_t + \phi_q \hat{Q}_t, \quad \text{with} \;\; \phi_q \equiv \phi_y \zeta \tag{B.16}$$

as output gap $\hat{y}_t$ is positively correlated with the asset price gap $\hat{Q}_t$ from (B.12). (B.16) is the policy reaction function that targets asset price $\hat{Q}_t$ as well as inflation π_t. Bernanke and Gertler (2000), by incorporating stochastic ad-hoc bubble components to asset prices in a model based on financial frictions à la Bernanke et al. (1999), study whether the monetary policy rule that directly targets asset price as in (B.16) is an effective business cycle stabilizer. Their conclusion is such rules are undesirable as they deter real economic activities when bubbles appear and burst.[16] In contrast, our framework features no *irrational* asset price bubble: fluctuations in $\hat{Q}_t$ reflect the *rational expectations* about business cycle fluc-

[16]Recently, Galí (2021) introduces rational bubbles in a New-Keynesian model with overlapping generations, arguing 'leaning against the bubble' monetary policy, if properly specified, can insulate the economy from the aggregate fluctuations due to rational bubbles.

tuations, and thus from monetary authority's perspective, targeting the stock price gap $\hat{Q}_t$ becomes equivalent to targeting the output gap $\hat{y}_t$, as the two gaps are perfectly correlated up to a first-order. Thus in our framework, a conventional monetary policy rule becomes equivalent to the rule of Bernanke and Gertler (2000).

As we did in Section 2, now we study whether equation (B.16) achieves divine coincidence as in textbook New-Keynesian models. Our objective now is to show that (i) this rule cannot guarantee equilibrium determinacy even if it satisfies the so-called Taylor principle; (ii) the aggregate financial volatility σ_t^q can be created in a self-fulfilling way as in Section 2. We first define $\phi \equiv \phi_q + \frac{\kappa(\phi_\pi - 1)}{\rho} > 0$, which is the total responsiveness of monetary policy to inflation and asset price gap. $\phi > 0$ corresponds to the conventional Taylor principle that guarantees the uniqueness of equilibriun in log-linearized models. Plugging (B.18) into (B.13), we obtain

$$d\hat{Q}_t = \left((\phi_\pi - 1)\pi_t + \phi_q \hat{Q}_t \underbrace{- \frac{\sigma^2}{2} + \frac{(\sigma + \sigma_t^q)^2}{2}}_{\text{New terms}} \right) dt + \sigma_t^q dZ_t. \tag{B.17}$$

Multiple equilibria Omitting the volatility feedback terms in the above (B.17), we obtain the usual log-linearized version of the $\hat{Q}_t$ dynamics as

$$d\hat{Q}_t = \left((\phi_\pi - 1)\, \pi_t + \phi_q \hat{Q}_t \right) dt + \sigma_t^q dZ_t,$$

with which the Taylor principle $\phi > 0$ ensures that we achieve the famous *divine coincidence*: $\hat{Q}_t = \pi_t = 0$ for $\forall t$ is the unique possible rational expectations equilibrium from Blanchard and Kahn (1980). In contrast, now that the aggregate financial volatility σ_t^q affects the drift of equation (B.17), we have multiple equilibria, and σ_t^q can possibly appear in a self-fulfilling way. The reason is similar to why we have self-fulfilling endogenous volatility in Section 2, i.e., σ_t^s.[17] Here, the dynamic IS equation (B.17) features the countercyclical financial volatility σ_t^q: an increase in σ_t^q raises the risk-premium. It in turn brings down the financial wealth and aggregate demand, thus raising the drift of (B.17).

Here is an intuitive way to think about the core reason why the financial volatility σ_t^q is created in a self-fulfilling manner. Imagine that capitalists in our model suddenly fear of a potential financial crisis that features higher levels of risk-premium and financial volatility:

[17] Due to the isomorphic mathematical structure between the dynamics in (B.13) and (8), we easily predict that σ_t^q can arise similarly to the ways σ_t^s arise in a self-fulfilling way in Section 2.

they respond by reducing their portfolio demand for the stock market, which leads to the collapse of the asset price, and self-justifies a higher expected return in the stock market and a rise in risk-premium. This result is related to Acharya and Dogra (2020)'s findings about equilibrium determinacy in models with countercyclical income risks, even though their paper focuses on *idiosyncratic* risks and the effects from precautionary savings, while ours centers on the alternative equilibria stemming from self-fulfilling *aggregate* endogenous risk.

We now formalize the multiple equilibrium intuition presented above by constructing a rational expectations equilibrium that supports an initial volatility σ_0^q. For simplicity, we focus on the case in which σ_0^q jumps off from $\sigma_0^{q,n} = 0$ (i.e., $\sigma_0^q > 0$).

Martingale equilibrium As in Section 2, we study one particular form of rational expectations equilibrium that supports an initial volatility σ_0^q: the equilibrium in which the asset price gap $\hat{Q}_t$ follows a martingale after σ_0^q appears. As $\hat{Q}_t$ is martingale, we obtain

$$\pi_t = \kappa \int_t^{\infty} e^{\rho(s-t)} \underbrace{\mathbb{E}_t(\hat{Q}_s)}_{=\hat{Q}_t} ds = \frac{\kappa}{\rho}\hat{Q}_t, \tag{B.18}$$

for π_t by iterating (B.15) over time, which implies that inflation π_t closely follows the trajectory of $\hat{Q}_t$. Plugging (B.18) into (B.17) and imposing the martingale condition, we obtain

$$\hat{Q}_t = -\frac{(\sigma + \sigma_t^q)^2}{2\phi} + \frac{\sigma^2}{2\phi}. \tag{B.19}$$

Our martingale equilibrium trajectory does not diverge on expectation in the long-run, as $\{\hat{Q}_t, \pi_t\}$ paths stay, on expectation, at their initial values, thus satisfying $\mathbb{E}_0(\pi_t) = \pi_0$ and $\mathbb{E}_0\left(\hat{Q}_t\right) = \hat{Q}_0$, $\forall t \geq 0$. The last step is to show that there exists a stochastic path of $\{\sigma_t^q\}$ starting from σ_0^q that supports this equilibrium. This equilibrium then both (i) supports an initial volatility $\sigma_0^q > 0$ and (ii) does not diverge in the long-run. Using equations (B.17) and (B.19),[18] we obtain the stochastic process of σ_t^q as given by

$$d\sigma_t^q = -\frac{\phi^2 \left(\sigma_t^q\right)^2}{2\left(\sigma + \sigma_t^q\right)^3} dt - \phi \frac{\sigma_t^q}{\sigma + \sigma_t^q} dZ_t. \tag{B.20}$$

[18] Since $\hat{Q}_t$ process is a martingale, the drift part in equation (B.17) must be 0 almost surely.

Both (B.19) and (B.20) constitute the dynamics of $\{\hat{Q}_t, \pi_t, \sigma_t^q\}$ in this particular rational expectations equilibrium supporting $\sigma_0^q > 0$. What does this equilibrium look like? Proposition B.4, like Proposition 1 of Section 2, sheds light on the behavior of $\{\hat{Q}_t, \pi_t, \sigma_t^q\}$ paths and argues that similarly to Section 2, $\{\hat{Q}_t, \pi_t, \sigma_t^q\}$ almost surely converge to a perfectly stabilized path (i.e., $\hat{Q}_t = \pi_t = \sigma_t^q = 0$) in the long run. Few paths that do not converge blow up asymptotically with vanishing probability and together with the forward-looking nature of the economy, help sustain the initial crisis.

Proposition B.4 (Bernanke and Gertler (2000) Rule and Indeterminacy) *For any value of Taylor responsiveness $\phi > 0$:*

1. Indeterminacy: *there is always a rational expectations equilibrium (REE) that supports initial volatility $\sigma_0^q > 0$ and is represented by $\hat{Q}_t$ and π_t dynamics in (B.19), and σ_t^q process in (B.20).*

2. Properties*: the equilibrium that supports an initial volatility $\sigma_0^q > 0$ satisfies:*

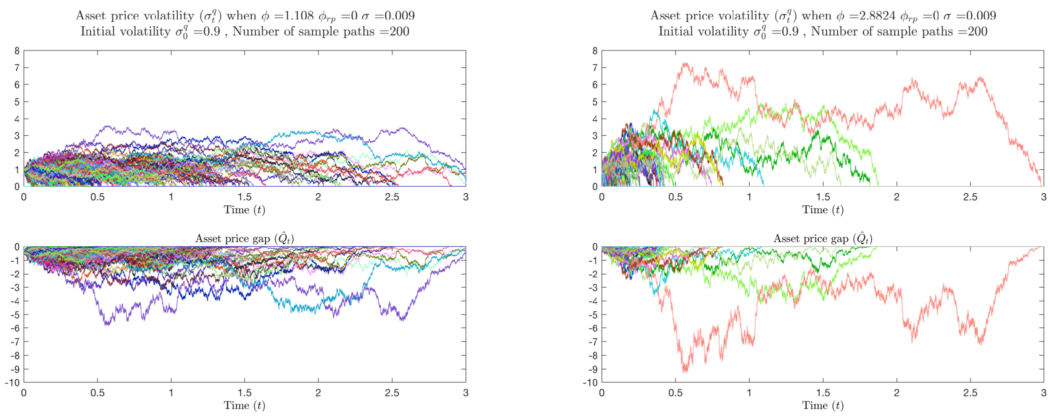
$$(i)\ \sigma_t^q \overset{a.s}{\to} \sigma_\infty^q = \sigma^{q,n} = 0,\ \ (ii)\ \hat{Q}_t \overset{a.s}{\to} 0 \text{ and } \pi_t \overset{a.s}{\to} 0,\ \text{ and }\ (iii)\ \mathbb{E}_0\left(\max_t(\sigma_t^q)^2\right) = \infty.$$

Proposition B.4 is similar to Proposition 1 due to the isomorphic equilibrium structure between Section 2 and Online Appendix B.5.[19] The conditions $\sigma_t^q \overset{a.s}{\to} \sigma_\infty^q = \sigma^{q,n} = 0$, $\hat{Q}_t \overset{a.s}{\to} 0$, and $\pi_t \overset{a.s}{\to} 0$ imply that equilibrium paths supporting an initial volatility $\sigma_0^q > 0$ are almost surely stabilized in the long run. Then, how is it possible for $\sigma_0^q > 0$ to appear at first? The finding $\mathbb{E}_0(\max_t(\sigma_t^q)^2) = \infty$ implies that an initial self-fulfilling shock σ_0^q and the ensuing crisis can be sustained by the vanishing probability of an ∞-severe financial disruption in the long future. This result has similar implications to Martin (2012) in a sense that our framework does not assume the existence of specific disasters but disaster risk is always present even if monetary authority satisfies the Taylor principle and actively stabilizes the business cycle. Martin (2012) applied a similar logic to asset pricing contexts and showed that the pricing of a broad class of long-dated assets is driven by the possibility of extraordinarily bad news in the future.

Calibration and Simulation For the rest of the paper, we calibrate our model parameters based on values commonly found in the previous literature: see Table D.1 in Appendix D

[19] Even with the presence of nontrivial inflation π_t, Figure 1 illustrates the construction of the martingale equilibrium in Proposition B.4.

for details. A few points are worth mentioning. For worker's risk-aversion parameter φ, we use $\varphi = 0.2$ following Gandelman and Hernández-Murillo (2014).[20] For an individual firm's labor share in production, we use $1 - \alpha = 0.6$ following Alvarez-Cuadrado et al. (2018), as we regard the aggregate labor in the production function as a proxy for the capital in conventional macroeconomic models. With this calibration, our co-movement condition (i.e., Assumption B.1) is satisfied.



(a) With $\phi_\pi = 1.5$

(b) With $\phi_\pi = 2.5$.

Figure B.1: $\{\sigma_t^q, \hat{Q}_t\}$ dynamics when $\sigma^{q,n} = 0$ and $\sigma_0^q = 0.9$

Figure B.1 illustrates the martingale equilibrium's dynamic paths of $\{\sigma_t^q, \hat{Q}_t\}$ supporting $\sigma_0^q = 0.9 > \sigma^{q,n} = 0$. Normalization shows that as σ_0^q jumps off by σ, stock price falls by $2 - 10\%$, which is consistent with our empirical findings in Online Appendix C.

Figure B.1 also explores the effects on the martingale equilibrium paths of a change in policy responsiveness to inflation ϕ_π. The right panel B.1b uses the default calibration value $\phi_\pi = 2.5$, while the left panel B.1a assumes a more accomodating stance $\phi_\pi = 1.5$. As we raise ϕ_π, we observe that sample paths are likely to converge faster towards full stabilization at the expense of an increased likelihood of a more severe crisis path in a given period of time. The intuition is simple: for a *given* level of initial volatility $\sigma_0^q > 0$ to be sustained under a more responsive policy rate with higher ϕ_π, it must be true that more amplified endogenous volatility (i.e., high σ_t^q) and severe recession (i.e., low $\hat{Q}_t$) arise with vanishing probability in the future.

[20] Gandelman and Hernández-Murillo (2014)'s estimates of φ range between 0.2 and 10. In our environment, a higher risk aversion of workers makes their labor supply (and therefore, output) less responsive to business cycle fluctuations. In that scenario, a higher asset price tends to translate into less wage income distributed to workers, making it harder to satisfy the co-movement condition (i.e., Assumption B.1). Thus, we pick a value on the lower end of the acceptable range and set $\varphi = 0.2$.

Booms In an analogous way, we can construct a rational expectations equilibrium that supports a negative volatility $\sigma_0^q < \sigma_t^{q,n} \equiv 0$. The equilibrium paths feature a boom phase with buoyant production and consumption and with lower levels of financial volatility and risk-premium. A higher ϕ value speeds up the stabilization process, but increases the likelihood of an equilibrium path with an overheated economy.[21]

B.6 Modified Monetary Rule

A modified monetary policy rule includes risk-premium as a separate factor as in

$$i_t = \underbrace{r^n + \phi_\pi \pi_t + \phi_q \hat{Q}_t}_{\text{Bernanke and Gertler (2000)}} - \underbrace{\frac{1}{2}\hat{rp}_t}_{\text{Risk-premium targeting}}, \quad \text{where } \hat{rp}_t \equiv \mathrm{rp}_t - \mathrm{rp}^n. \tag{B.21}$$

The above monetary policy rule in (B.21) contains a 'risk-premium gap term' as a factor in addition to inflation and asset price gap. It also can be written in terms of the risk-adjusted natural rate r_t^T as

$$i_t = r_t^T + \phi_\pi \pi_t + \phi_q \hat{Q}_t,$$

where a higher $\hat{rp}_t$ brings down r_t^T, forcing i_t to fall. The following Proposition B.5 establishes that a monetary policy rule following (B.21) and that satisfies the Taylor principle, i.e., $\phi > 0$ restores equilibrium determinacy and fully stabilizes the economy.

Proposition B.5 (Risk-Premium Targeting and Ultra-Divine Coincidence) *The monetary policy rule*

$$i_t = r^n + \phi_\pi \pi_t + \phi_q \hat{Q}_t - \frac{1}{2}\hat{rp}_t, \quad \textit{where} \quad \phi \equiv \phi_q + \frac{\kappa(\phi_\pi - 1)}{\rho} > 0, \tag{B.22}$$

achieves $\hat{Q}_t = \pi_t = \hat{rp}_t = 0$ as unique rational expectations equilibrium (REE). Therefore, the monetary policy rule in (B.22) attains stabilization in (i) output and asset price, (ii) inflation, and (iii) financial market (i.e., financial volatility and risk-premium). We call it the ultra-divine coincidence.

This result is a direct consequence of Blanchard and Kahn (1980) and Buiter (1984). The reason that central banks target risk-premium as a separate factor is that this term directly appears in the drift of our dynamic IS equation (i.e., (B.13)). According to the policy

[21] Two singular points exist in the $\{\sigma_t^q\}$ process in (B.20): as σ_t^q hits $-\sigma$, both drift and volatility diverge, and $\{\sigma_t^q\}$ process features a jump. When σ_t^q hits 0, it stays there forever so $\sigma_t^q = 0$ thereafter.

rule in (B.22), central banks lower the policy rate i_t when $\text{rp}_t > \text{rp}^n$ to boost $\hat{Q}_t$ and $\hat{C}_t$,[22] since a higher risk-premium drags down asset price and business cycle levels. If monetary policy offsets effects of the excess volatility (or excess risk-premium) with this additional target in its rule, it precludes the possibility of self-fulfilling rises in financial volatility. Combined with the Taylor principle (i.e., $\phi > 0$) that guarantees unique equilibrium in a log-linearized setting, the policy rule in equation (B.22) restores equilibrium determinacy and achieves both macro stability (with $\hat{Q}_t = \pi_t = 0$) and financial stability (with $\hat{rp}_t = 0$, which implies $\text{rp}_t = \text{rp}^n$ and $\sigma_t^q = \sigma_t^{q,n} = 0$). The interest rate on the equilibrium path then becomes $i_t = r^n$, which is the same level as in the equilibrium path of a canonical New-Keynesian model. Therefore, the ultra-divine coincidence result implies: one policy tool (i_t rule) achieves an additional objective (financial stability) in addition to the two usual mandates (output gap and inflation stability). This is possible in our framework because financial markets and the business cycle are tightly interwoven and real and financial instabilities are equivalent to each other.

A common view holds that monetary policy should respond to financial market disruptions only when they affect (or to the degree that they affect) the original mandates (i.e., inflation stability and full employment). This premise is at odds with our results: the failure to target the risk-premium of financial markets subjects the economy to the apparition of self-fulfilling financial volatility and risk-premium, and the corresponding recessions and overheating episodes that ensue. Only by targeting risk-premium in the particular way characterized in (B.21), the monetary authority can re-establish equilibrium determinacy and achieve the ultra-divine coincidence outlined in the previous paragraphs.

Interpretation We can rewrite our modified Taylor rule in (B.22) as

$$\underbrace{i_t + \text{rp}_t}_{=i_t^m} - \underbrace{\frac{1}{2}\text{rp}_t}_{\text{Ito term}} = \underbrace{r^n + \text{rp}^n}_{=i_t^{m,n}} - \underbrace{\frac{1}{2}\text{rp}^n}_{\text{Ito term}} + \underbrace{\phi_\pi \pi_t + \phi_q \hat{Q}_t}_{\text{Business cycle targeting}},$$

or equivalently as

[22] Even with Bernanke and Gertler (2000) rule, monetary policy responds to a rise in risk-premium since it negatively affects the asset price gap $\hat{Q}_t$ and inflation π_t. Our point is that the policy rate must systematically respond to deviations of rp_t from its natural level rp^n given $\hat{Q}_t$ and π_t levels.

$$\underbrace{\underbrace{\rho}_{\substack{\text{Dividend}\\ \text{yield}}} + \underbrace{\frac{\mathbb{E}_t\left(d\log a_t\right)}{dt}}_{\substack{\text{Internal rate of return}\\ \text{of aggregate wealth}}}}_{\text{Cum-dividend stock return}} = \underbrace{\underbrace{\rho}_{\substack{\text{Dividend}\\ \text{yield}}} + \frac{\mathbb{E}_t\left(d\log a_t^n\right)}{dt}}_{\text{Benchmark cum-dividend stock return}} + \underbrace{\phi_\pi \pi_t + \phi_q \hat{Q}_t}_{\text{Business cycle targeting}}, \tag{B.23}$$

where a_t is the economy's aggregate financial wealth, i.e., $p_t A_t Q_t$, and a_t^n is the aggregate wealth of the natural (i.e., flexible price) economy. Our modified monetary policy rule that targets a risk-premium as prescribed in equation (B.22) thus can be interpreted as a rule on the rate of change of $\log$-aggregate wealth as a function of traditional inflation and output gap (asset price) targets. Basically, the rate that determines the households' intertemporal substitution should be the expected return on stock markets, instead of just the risk-free policy rate, and therefore in order to achieve determinacy as well as stabilization in our model, the expected return on stock markets must target business cycles.

We interpret the rule in (B.23) as the *generalized Taylor rule*. With this rule, the central bank uses the aggregate wealth and its rate of return as *intermediate* targets towards achieving business cycle stabilization.

Practicality Some issues still remain about the feasibility to implement this new policy rule in (B.22). First, the risk premium gap $\hat{\text{rp}}_t$ in (B.21) depends on the natural level of risk-premium, rp^n, which is a counterfactual variable by definition, and therefore its observation is subject to some form of measurement error. Second, there are multiple kinds of risk-premia in financial markets that can be possibly targeted through monetary policy, and the construction of the aggregate risk-premium index as featured in our model might be subject to error as well. More importantly, and related to the previous two points, the coefficient attached to the risk-premium in (B.21) is exactly $\frac{1}{2}$. Given the potential for measurement error in $\hat{rp}_t$, it might be impossible for the central bank to target the risk-premium with the exact strength demanded by (B.21).[23] To understand the consequences of deviating from the $\frac{1}{2}$ risk-premium target, we consider the following alternative rule:

$$i_t = r^n + \phi_\pi \pi_t + \phi_q \hat{Q}_t - \phi_{\text{rp}} \hat{rp}_t, \tag{B.24}$$

[23] As an example, consider a multiplicative measurement error ε_t such that $\hat{rp}_t^{obs} = \varepsilon_t \cdot \hat{rp}_t$, where $\hat{rp}_t^{obs}$ is the measured premium.

where ϕ_{rp} is a constant term potentially different from $\frac{1}{2}$. With the policy rule in (B.24), we obtain

$$d\hat{Q}_t = \left((\phi_\pi - 1)\pi_t + \phi_q \hat{Q}_t + \left(\frac{1}{2} - \phi_{\text{rp}} \right) \hat{rp}_t \right) dt + \sigma_t^q dZ_t. \tag{B.25}$$

as $\{\hat{Q}_t\}$ dynamics. With $\phi_{\text{rp}} = \frac{1}{2}$, we return to determinacy (i.e., Proposition B.5). With $\phi_{\text{rp}} \neq \frac{1}{2}$, the martingale equilibrium with self-fulfilling volatility σ_t^q reappears and is characterized by[24]

$$\hat{Q}_t = -\frac{(\sigma + \sigma_t^q)^2}{2\phi_{\text{total}}} + \frac{\sigma^2}{2\phi_{\text{total}}} \quad \text{with} \quad \phi_{\text{total}} \equiv \frac{\phi}{1 - 2\phi_{\text{rp}}}, \tag{B.26}$$

where $\{\sigma_t^q\}$'s stochastic process after an initial volatility σ_0^q appears is given as

$$d\sigma_t^q = -\frac{\phi_{\text{total}}^2 \left(\sigma_t^q\right)^2}{2\left(\sigma + \sigma_t^q\right)^3} dt - \phi_{\text{total}} \frac{\sigma_t^q}{\sigma + \sigma_t^q} dZ_t. \tag{B.27}$$

When $\phi_{\text{rp}} < \frac{1}{2}$, including the case of $\phi_{\text{rp}} = 0$ in Proposition B.4, an increase in ϕ_{rp} leads to an increase in ϕ_{total} from (B.26). From (B.27), we observe that a higher ϕ_{total} accelerates the convergence of sample paths while creating more amplified ones given initial volatility σ_0^q. As far as $\phi_{\text{rp}} < \frac{1}{2}$, a higher ϕ_{rp} means monetary policy responds more strongly to fluctuations in $\hat{rp}_t$, which allows for faster stabilization. As ϕ_{rp} goes up from 0 to $\frac{1}{2}$, fluctuations in $\hat{rp}_t$ have a weaker direct effect on the drift of (B.25). Thus, the volatility of $\{\sigma_t^q\}$ process in (B.27) must rise to ensure that the initial volatility σ_0^q is supported, as on average the economy is better stabilized with a higher ϕ_{rp}. $\{\hat{Q}_t\}$ eventually is stabilized, which results, on average, on shorter but more amplified sample paths.

$\phi_{\text{rp}} < 0$ case is interesting since it implies central bank raises the policy rate in response to an increase of the risk premia. It is consistent with the *Real Bills Doctrine* which was a popular idea during the first half of the 20th century. Basically, the doctrine advocated for the Fed discount rate to track the average interest rate of the financial markets, as a means of stabilization. In our framework, $\phi_{\text{rp}} < 0$ pushes down ϕ_{total} from ϕ, thereby effectively slowing down the pace of stabilization after self-fulfilling σ_0^q arises. So this confirms that the *Real Bills Doctrine* with $\phi_{\text{rp}} < 0$ is not suitable for stabilization purposes, as empirically documented by Richardson and Troost (2009).

With $\phi_{\text{rp}} > \frac{1}{2}$, monetary policy responds too strongly to fluctuations in risk-premium, thus with an initial positive volatility $\sigma_0^q > 0$, the policy rate drops too excessively and

[24] Equations (B.24) and (B.26) are easily derived in a similar way to Proposition B.4.

creates an artificial boom instead of a crisis.[25] A higher ϕ_{rp} reduces $|\phi_{\text{total}}|$ and slows down stabilization since a higher ϕ_{rp} means monetary policy deviates more from determinacy (the case of $\phi_{\text{rp}} = \frac{1}{2}$), and thus gets less efficient at stabilization. Table D.3 and Figure D.7 in Online Appendix D summarize our discussion and provides simulation results, respectively.

C Suggestive Evidence

Purpose of this section In this section, we provide the empirical evidence that financial volatility is an important driver of the business cycle. The impulse-response function results in this section provides moments to match with our model with stock markets in Online Appendix B, as we saw in Figure B.1.

Stock market volatility is commonly viewed in the literature as a proxy of financial and economic uncertainty, which Bloom (2009) and later Gilchrist and Zakrajšek (2012), Bachmann et al. (2013), Jurado et al. (2015), Caldara et al. (2016), Baker et al. (2020), Coibion et al. (2021) further studied as a driving force behind business cycles fluctuations. In this Section, we evaluate these claims and present interesting empirical results. Figure D.4 in Online Appendix D provides the first piece of supportive evidence in that direction. Panel D.4a depicts several variables commonly used in the literature to measure financial uncertainty. The correlation between series is remarkably high and they all display positive spikes at the beginning and/or initial months following an NBER-dated recession, which is consistent with the evidence that many of these episodes were financial in nature.[26] Panel D.4b plots Ludvigson et al. (2021) (henceforth, LMN) financial and real (i.e. non-financial) uncertainty series. These variables are positively correlated and display a similar propensity to increase around recessions, though a different type of crisis (e.g. financial or not) is correlated with a different type of uncertainty playing the dominant role. For example, the massive spike in real vis-à-vis financial uncertainty following the recent Covid-19 recession, which initially was a health crisis that spilled into the real economy, can be observed in Panel D.4b.

The patterns displayed in Figure D.4 do not yet constitute a proof of the importance of financial market uncertainty as a driver of the business cycle, as we should worry about the possibility of reverse causation running from unfavorable economic conditions towards

[25] With $\phi_{\text{rp}} > \frac{1}{2}$, $\phi_{\text{total}} < 0$ from (B.26). Therefore $\sigma_t^q > 0$ is equivalent to the boom phase with $\pi_t > 0$ and $\hat{Q}_t > 0$.

[26] See Reinhart and Rogoff (2009) and Romer and Romer (2017) for the classification of the past recessions. Their analysis showed many recessions had roots in financial markets.

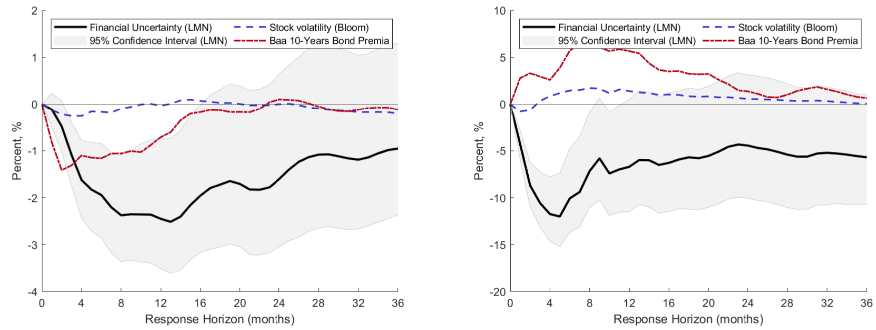
uncertainty. We tackle this issue by proposing a simple Vector Autoregression (VAR) with the structural identification strategy based on the timing of macroeconomic shocks similar to Bloom (2009). Equation (C.1) presents the variables considered and their ordering, with non-financial series first and financial variables last.[27]

$$\text{VAR-11 order:} \quad \begin{bmatrix} \log(\text{Industrial Production}) \\ \log(\text{Employment}) \\ \log(\text{Real Consumption}) \\ \log(\text{CPI}) \\ \log(\text{Wages}) \\ \text{Hours} \\ \text{Real Uncertainty (LMN)} \\ \text{Fed Funds Rate} \\ \log(\text{M2}) \\ \log(\text{S\&P-500 Index}) \\ \text{Financial Uncertainty (LMN)} \end{bmatrix} \tag{C.1}$$

Both LMN real and financial uncertainty measures are included to differentiate the effects of financial volatility shocks from the effects from real uncertainty. For similar reasons, we include the S&P-500 index in our VAR to empirically distinguish between shocks affecting the level of financial markets and shocks affecting their volatility. In order to ameliorate possible concerns about the validity of the structural identification strategy, we estimate our VAR using monthly data, where the identification assumptions are more likely to hold. Figure C.2 presents the impulse responses to the orthogonalized financial uncertainty shock. Panel C.2a plots the response of industrial production, which falls by up to 2.5% and displays moderate persistence following a one standard deviation shock to financial uncertainty. Panel C.2b plots the response of the S&P-500 Index, which drops up to 12% within the first four months before gradually recovering. Together, both pictures imply a rise of financial uncertainty depresses both industrial activity and financial markets.

Figure C.2 also features alternative estimates using common financial uncertainty proxies such as Bloom (2009) stock market volatility index and 10-years premium on Baa-rated

[27] The ordering is used by Ludvigson et al. (2021), who, using identification strategy based on event constraints, find that the uncertainty of financial markets tends to be an exogenous source of business cycle fluctuations, while the real uncertainty is more likely an endogenous response to the business cycle fluctuations. We also have implemented alternative specifications and orderings that produced qualitatively similar results (not reported, provided upon request).



(a) Response: Industrial Production (b) Response: S&P-500 Index

Figure C.2: Impulse Response Functions (IRFs), selected series. Figures C.2a and C.2b display the response to a one standard deviation financial uncertainty shock of monthly (log) Industrial Production and (log) S&P-500 Index series, respectively, using a VAR-11 with the variable composition and ordering given in (C.1). Shaded area indicates 95% confidence interval around financial uncertainty measure computed using standard bootstrap techniques.

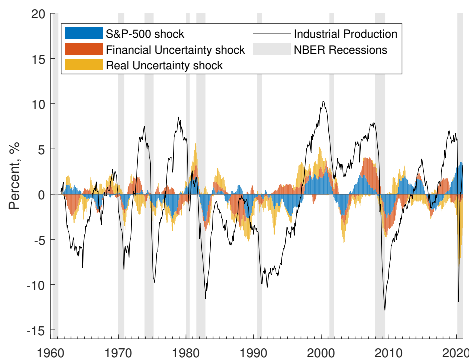
corporate bonds. The responses are generally more muted, and take the opposite sign in the case of the S&P Index. These results can be explained by the fact that standard proxies contain information unrelated to financial uncertainty that distorts our estimates (see Jurado et al. (2015) for a discussion), and therefore we choose LMN as our preferred financial uncertainty measure. In Online Appendix D, we report additional impulse response estimates. Especially, the Figure D.6 in Online Appendix D shows that monetary authorities respond with accommodating interest rate movements to financial uncertainty shocks, while real uncertainty has no statistically significant effect on either interest rates or stock market fluctuations.

Finally, we can further explore the contribution of financial uncertainty to business cycle fluctuations by looking at Table D.2 in Online Appendix D, which reports the Forecast Error Variance Decomposition (FEVD) of Industrial Production and the S&P-500 Index. Financial uncertainty shocks explain close to 5% of the fluctuations in both series, while real uncertainty explains an additional 2-4% of movements in industrial activity in the medium run. Figure D.3 provides a more graphical illustration of these results by plotting the historical decomposition of the series. We observe that the contribution of financial uncertainty rivals that of shocks to the level of financial variables captured by the S&P-500 shock, and is especially important in driving industrial production boom-bust patterns during and in the preceding months of recessionary episodes.

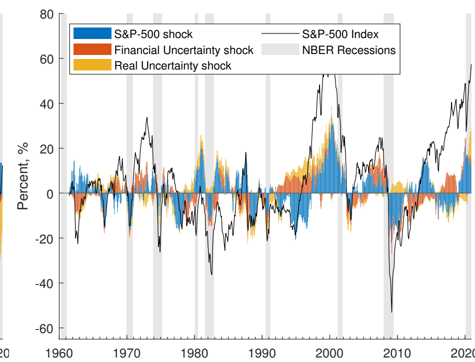
D Additional Figures and Tables

Parameter	Value	Description
φ	0.2	Relative Risk Aversion
χ_0	0.25	Inverse Frisch labor supply elasticity
ρ	0.020	Subjective time discount factor
σ	0.0090	TFP volatility
g	0.0083	TFP growth rate
α	0.4	$1 -$ Labor income share
ϵ	7	Elasticity of substitution intermediate goods
δ	0.45	Calvo price resetting probability
ϕ_π	2.50	Policy rule inflation response
ϕ_y	0.11	Policy rule output gap response
ϕ_{rp}	0	Policy rule risk premium response
$\bar{\pi}$	0	Steady state trend inflation target

Table D.1: Baseline parameter calibration used in Online Appendix B.5

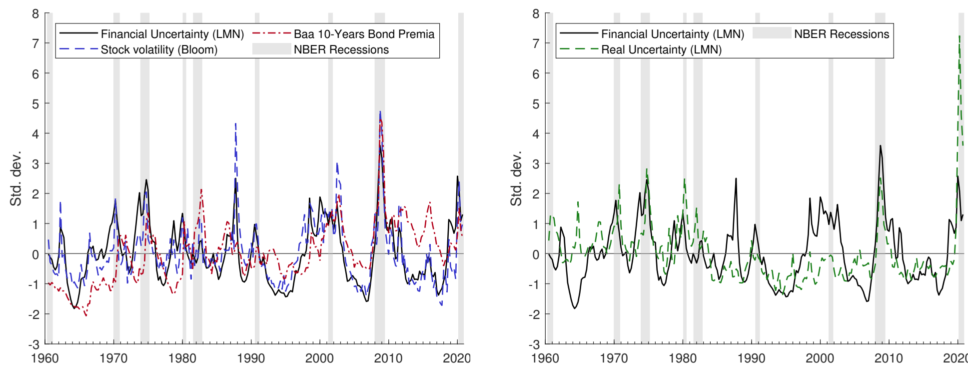


(a) Industrial Production



(b) S&P-500 Index

Figure D.3: Historical Decomposition, selected series. Figures D.3a and D.3b display the historical decomposition of monthly Industrial Production and S&P-500 Index series, respectively, based on the VAR-11 with variable composition and ordering in (C.1). Variables are de-trended by subtracting the contribution of initial conditions and constant terms after series decomposition. Columns report a contribution of each shock to the fluctuations around trend of the variable considered.



(a) Financial Uncertainty series

(b) Financial vs. Real Uncertainty

Figure D.4: Uncertainty series. Figure D.4a displays common measures of financial uncertainty. Figure D.4b displays Ludvigson et al. (2021) (henceforth, LMN) measures of financial and real economic uncertainty. LMN financial and real economic uncertainty series are constructed as the average volatility of the residuals from predictive regressions on financial and real economic variables, respectively (See Ludvigson et al. (2021)). Bloom (2009)'s stock market volatility is constructed using VXO data from 1987 onward and the monthly volatility of the S&P 500 index normalized to the same mean and variance in the overlapping interval for the 1960-1987 period (See Bloom (2009)). The bond risk-premia series is the Moody's seasoned Baa corporate bond yield relative to the yield on a 10-year treasury bond at constant maturity. The depicted series have a normalized zero mean and one standard deviation.

(i) Industrial Production

Horizon	Fin. Uncert. (LMN)	Real Uncert. (LMN)	Stock Vol. (Bloom)	Baa 10-Yr Premia
h=1	0	0.30	0.21	0.12
h=6	1.27	3.37	2.98	1.36
h=12	4.28	4.38	3.16	1.94
h=36	3.24	1.67	1.98	0.64

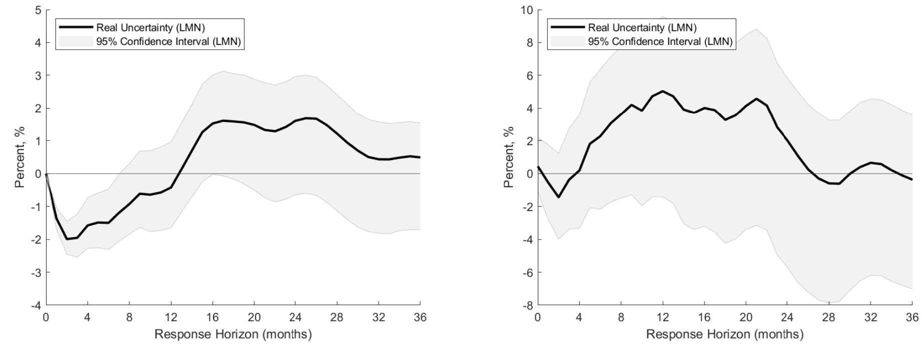
(ii) S&P-500 Index

Horizon	Fin. Uncert. (LMN)	Real Uncert. (LMN)	Stock Vol. (Bloom)	Baa 10-Yr Premia
h=1	0.11	0.08	0.39	0.06
h=6	3.30	0.25	3.26	0.62
h=12	4.77	0.54	10.03	2.16
h=36	6.50	0.91	12.16	2.40

(iii) Fed Funds Rate

Horizon	Fin. Uncert. (LMN)	Real Uncert. (LMN)	Stock Vol. (Bloom)	Baa 10-Yr Premia
h=1	0.01	0.98	0	0.08
h=6	0.42	0.84	3.11	1.66
h=12	1.47	0.91	4.69	2.30
h=36	2.81	2.05	5.02	3.17

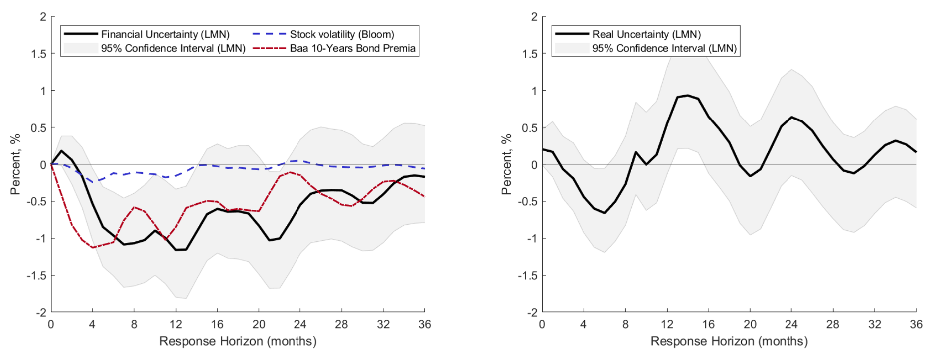
Table D.2: Forecast Error Variance Decomposition (FEVD). The table presents the variance contribution (in percentage) of financial and real uncertainty shocks to selected series at different time horizons (in months). The FEVD is constructed using a VAR-11 with equation (C.1) variable composition and ordering. The first two columns report the contribution of LMN financial and real uncertainty shocks, respectively. The last two columns report alternative VAR specifications where the preferred LMN financial uncertainty measure (column one) is replaced by common proxies employed in the literature, either Bloom (2009) stock market volatility measure or the Baa 10-years corporate bond premia, respectively.



(a) Response: Industrial Production

(b) Response: S&P-500 Index

Figure D.5: Impulse Response Functions (IRFs), selected series. Figures D.5a and D.5b display the response to one standard deviation real uncertainty shock by monthly (log) Industrial Production and (log) S&P-500 Index series, respectively, using a VAR-11 with equation (C.1) variable composition and ordering. Shaded area indicates 95% confidence interval around preferred financial uncertainty measure computed using standard bootstrap techniques.



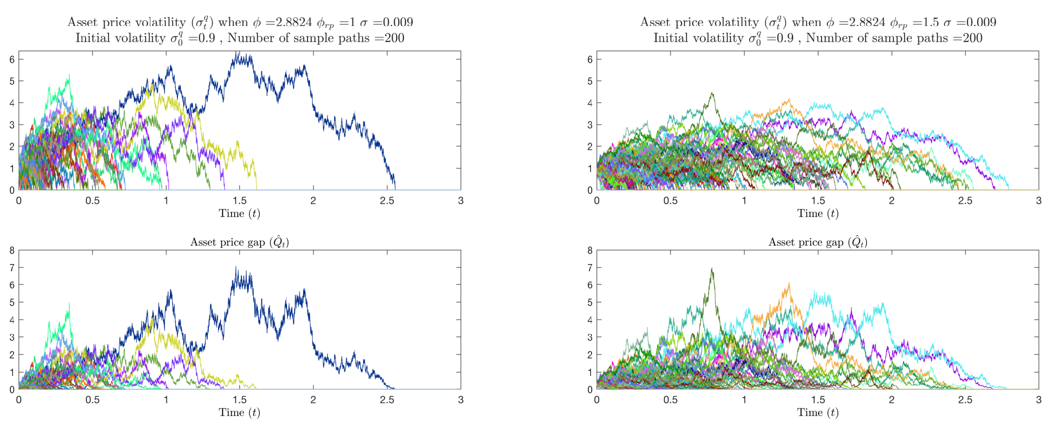
(a) Shock: Financial Uncertainty

(b) Shock: Real Uncertainty

Figure D.6: Impulse Response Functions (IRFs), Fed Funds Rate. This Figure displays the response to a one standard deviation uncertainty (financial or real) shock by monthly Fed Funds Rate series, using a VAR-11 with equation (C.1) variable composition and ordering. Panel D.6a plots the response to a financial uncertainty shock, and Panel D.6b to a real uncertainty shock. Shaded area indicates 95% confidence interval around preferred financial/real uncertainty measure computed using standard bootstrap techniques. Additional lines display alternative impulse responses obtained by substituting preferred LMN financial uncertainty measure with common proxies employed in the literature.

$\phi_{\text{rp}} < 0$ **(Real Bills Doctrine)**	$0 \leq \phi_{\text{rp}} < \frac{1}{2}$
(i) With $\phi_{\text{rp}} \downarrow$, convergence speed$\downarrow$ and less amplified paths (ii) $\sigma_t^q > \sigma_t^{q,n} = 0$ means a crisis ($\hat{Q}_t < 0$ and $\pi_t < 0$)	(i) With $\phi_{\text{rp}} \uparrow$, convergence speed$\uparrow$ and more amplified paths (ii) $\sigma_t^q > \sigma_t^{q,n} = 0$ means a crisis ($\hat{Q}_t < 0$ and $\pi_t < 0$)
$\phi_{\text{rp}} = \frac{1}{2}$	$\phi_{\text{rp}} > \frac{1}{2}$
No sunspot (ultra-divine coincidence)	(i) With $\phi_{\text{rp}} \uparrow$, convergence speed$\downarrow$ and less amplified paths (ii) $\sigma_t^q > \sigma_t^{q,n} = 0$ means a boom ($\hat{Q}_t > 0$ and $\pi_t > 0$)
As $\phi \uparrow$, convergence speed$\uparrow$ and $\exists$ more amplified paths	

Table D.3: Effects of different parameters $\{\phi_{\text{rp}},\ \phi\}$ on stabilization in Section B.6



(a) With $\phi_{\text{rp}} = 1$

(b) With $\phi_{\text{rp}} = 1.5$.

Figure D.7: $\{\sigma_t^q, \hat{Q}_t\}$ dynamics when $\sigma^{q,n} = 0$ and $\sigma_0^q = 0.9$, with varying $\phi_{\text{rp}} > \frac{1}{2}$

E Derivations and Proofs for Online Appendix B

Worker's optimization At each time t, each hand-to-mouth worker solves

$$\max_{C_{W,t},N_{W,t}} \frac{\left(\frac{C_{W,t}}{A_t}\right)^{1-\varphi}}{1-\varphi} - \frac{(N_{W,t})^{1+\chi_0}}{1+\chi_0} \quad \text{s.t.} \quad p_t C_{W,t} = w_t N_{W,t}. \tag{E.1}$$

Solving (E.1) is trivial, resulting in

$$N_{W,t} = \left(\frac{w_t}{p_t}\right)^{\frac{1-\varphi}{\chi_0+\varphi}} \frac{1}{A_t^{\frac{1-\varphi}{\chi_0+\varphi}}} = \left(\frac{w_t}{p_t A_t}\right)^{\frac{1}{\chi}}, \quad C_{W,t} = \frac{w_t}{p_t} N_{W,t} = \left(\frac{w_t}{p_t}\right)^{1+\frac{1}{\chi}} A_t^{-\frac{1}{\chi}}, \tag{E.2}$$

where we use $\chi \equiv \frac{\chi_0+\varphi}{1-\varphi}$ in Definition B.1.

Capitalist's optimization In equilibrium, each capitalist chooses $\theta_t = 1$ as bond markets are in zero net supply. Using $\rho a_t = p_t C_t$ from (B.5), the capitalists' budget flow constraint in (B.3) becomes:

$$\frac{da_t}{a_t} = (i_t^m - \rho)\, dt + (\sigma + \sigma_t^q + \sigma_t^p)\, dZ_t. \tag{E.3}$$

The capitalist's state price density in equilibrium is thereby given by

$$\xi_t^N = e^{-\rho t} \frac{1}{p_t C_t} = e^{-\rho t} \frac{1}{\rho a_t}, \tag{E.4}$$

on which we can apply Ito's Lemma and obtain

$$\begin{aligned} -\frac{d\xi_t^N}{\xi_t^N} &= \frac{da_t}{a_t} - \left(\frac{da_t}{a_t}\right)^2 + \rho dt \\ &= \underbrace{\left(i_t^m - (\sigma + \sigma_t^q + \sigma_t^p)^2\right)}_{=i_t} dt + (\sigma + \sigma_t^q + \sigma_t^p)\, dZ_t = i_t dt + (\sigma + \sigma_t^q + \sigma_t^p) dZ_t \end{aligned} \tag{E.5}$$

with which we obtain $i_t + (\sigma + \sigma_t^q + \sigma_t^q)^2 = i_t^m$ (i.e., equation (B.6)) from $\mathbb{E}_t\left(-\frac{d\xi_t^N}{\xi_t^N}\right) = i_t dt$. Note that (B.5) and (E.5) are the same conditions as in Merton (1971).

Proof of Lemma B.1. We know that in equilibrium, each capitalist holds the financial wealth $a_t = p_t A_t Q_t$ since all of them are identical both ex-ante and ex-post. We start by stating capitalist's nominal state-price density ξ_t^N and real state-price density ξ_t^r. The nominal state-price density is relevant to the nominal interest rate, while the real state-price density matters when we calculate the real interest rate. The real state price density ξ_t^r is given by

$$\xi_t^r = e^{-\rho t}\frac{1}{C_t} = p_t \xi_t^N. \tag{E.6}$$

Using (E.5), we can apply Ito's Lemma to (E.6) and obtain

$$\frac{d\xi_t^r}{\xi_t^r} = \left(\underbrace{\pi_t - i_t - \sigma_t^p\left(\sigma + \sigma_t^q + \sigma_t^p\right)}_{=-r_t}\right)dt - (\sigma + \sigma_t^q)dZ_t, \tag{E.7}$$

from which we obtain the Fisher identity with the inflation premium in equation (B.9):

$$r_t = i_t - \pi_t + \sigma_t^p\left(\sigma + \sigma_t^q + \sigma_t^p\right). \tag{E.8}$$

■

Proof of Proposition B.1. We start from the intermediate firms' optimization problem. As we have the externality à la Baxter and King (1991), we need to go through additional steps in aggregaing individual decisions across firms. Let firm i take its demand function as given and choose the optimal price $p_t(i)$ at any t. With $E_t \equiv (N_{W,t})^{\alpha}$, from the production function, we have

$$n_t(i) = \left(\frac{y_t(i)}{A_t E_t}\right)^{\frac{1}{1-\alpha}}. \tag{E.9}$$

Then each firm i chooses p_i that maximizes its profit, solving

$$\max_{p_t(i)} p_t(i)\left(\frac{p_t(i)}{p_t}\right)^{-\epsilon} y_t - w_t\left(\frac{y_t}{A_t E_t}\right)^{\frac{1}{1-\alpha}}\left(\frac{p_t(i)}{p_t}\right)^{-\frac{\epsilon}{1-\alpha}}. \tag{E.10}$$

In the flexible price economy, all firms charge the same price (i.e., $p_t(i) = p_t \ \forall i$) and hire the same amount of labor (i.e., $n_t(i) = N_{w,t} \ \forall i$). From (E.10), we obtain

$$\frac{w_t^n}{p_t^n} = \frac{\epsilon-1}{\epsilon}(1-\alpha)y_t^{\frac{-\alpha}{1-\alpha}}\left(A_t\right)^{\frac{1}{1-\alpha}} N_{W,t}{}^{\frac{\alpha}{1-\alpha}} = \frac{\epsilon-1}{\epsilon}(1-\alpha)y_t^{\frac{-\alpha}{1-\alpha}}\left(A_t\right)^{\frac{1}{1-\alpha}}\left(\frac{w_t^n}{p_t^n}\right)^{\frac{\alpha}{\chi(1-\alpha)}} A_t^{\frac{-\alpha}{\chi(1-\alpha)}}, \tag{E.11}$$

from which we obtain the following equilibrium real wage:

$$\frac{w_t^n}{p_t^n} = \left(\frac{\epsilon-1}{\epsilon}(1-\alpha)\right)^{\frac{\chi(1-\alpha)}{\chi(1-\alpha)-\alpha}} y_t^{\frac{-\chi\alpha}{\chi(1-\alpha)-\alpha}} A_t^{\frac{\chi-\alpha}{\chi(1-\alpha)-\alpha}}. \tag{E.12}$$

In flexible price equilibrium, we know the aggregate production is linear, i.e., $y_t = A_t N_{W,t}$. Therefore, we obtain

$$y_t = A_t \left(\frac{\epsilon - 1}{\epsilon}(1-\alpha)\right)^{\frac{(1-\alpha)}{\chi(1-\alpha)-\alpha}} y_t^{\frac{-\alpha}{\chi(1-\alpha)-\alpha}} A_t^{\frac{1-\frac{\alpha}{\chi}}{\chi(1-\alpha)-\alpha}} A_t^{-\frac{1}{\chi}}. \tag{E.13}$$

From (E.13), we write the natural level of output y_t^n and the natural real wage $\frac{w_t^n}{p_t^n}$ as

$$y_t^n = \left(\frac{\epsilon - 1}{\epsilon}(1-\alpha)\right)^{\frac{1}{\chi}} A_t \text{ and } \frac{w_t^n}{p_t^n} = \frac{\epsilon - 1}{\epsilon}(1-\alpha)A_t, \tag{E.14}$$

from which in equilibrium, we obtain

$$N_{W,t}^n = \left(\frac{\epsilon - 1}{\epsilon}(1-\alpha)\right)^{\frac{1}{\chi}} \text{ and } C_{W,t}^n = \left(\frac{\epsilon - 1}{\epsilon}(1-\alpha)\right)^{1+\frac{1}{\chi}} A_t. \tag{E.15}$$

In equilibrium, consumption of capitalists and workers add up to the final output produced (i.e., equation (B.7)). Based on (E.15), we obtain

$$\rho A_t Q_t^n + \left(\frac{\epsilon - 1}{\epsilon}(1-\alpha)\right)^{1+\frac{1}{\chi}} A_t = \left(\frac{\epsilon - 1}{\epsilon}(1-\alpha)\right)^{\frac{1}{\chi}} A_t. \tag{E.16}$$

where we define Q_t^n to be the natural level of detrended stock price. Therefore we obtain Q_t^n and C_t^n, given by

$$Q_t^n = \frac{1}{\rho}\left(\frac{\epsilon - 1}{\epsilon}(1-\alpha)\right)^{\frac{1}{\chi}} \left(1 - \frac{(\epsilon-1)(1-\alpha)}{\epsilon}\right), \tag{E.17}$$

and $C_t^n = \rho A_t Q_t^n$. Since Q_t^n is constant, there is no drift and volatility for its process in the flexible price economy, thus we have $\mu_t^{q,n} = \sigma_t^{q,n} = 0$. To calculate the natural interest rate r_t^n, we start from the capital gain component in equation (B.8). By applying Ito's lemma, we obtain

$$\mathbb{E}\frac{d\left(p_t A_t Q_t\right)}{p_t A_t Q_t}\frac{1}{dt} = \pi_t + \underbrace{\mu_t^q}_{=0} + g + \underbrace{\sigma_t^q}_{=0} \sigma_t^p + \sigma \left(\sigma_t^p + \underbrace{\sigma_t^q}_{=0}\right). \tag{E.18}$$

As the dividend yield is always ρ, imposing expectation on both sides of (B.8) and combining with the equilibrium condition in equation (B.6) yields

$$i_t^m = \rho + \pi_t + g + \sigma\sigma_t^p = i_t + (\sigma + \sigma_t^p)^2 \,. \tag{E.19}$$

Plugging (E.19) to the real interest rate formula in Lemma B.1, we express the natural rate of interest r_t^n as

$$r_t^n = i_t - \pi_t + \sigma_t^p \left(\sigma + \underbrace{\sigma_t^{q,n}}_{=0} + \sigma_t^p \right) = \rho + g - \sigma^2, \tag{E.20}$$

which is a function of structural parameters including σ, proving (iii) of Proposition B.1. Since capitalists' consumption C_t^n is directly proportional to TFP A_t, we know

$$\frac{dC_t^n}{C_t^n} = gdt + \sigma dZ_t = \left(r_t^n - \rho + \sigma^2\right) dt + \sigma dZ_t, \tag{E.21}$$

where we use $r_t^n - \rho + \sigma^2 = g$ from equation (E.20).

■

Proof of Proposition B.2. In the sticky price equilibrium, we would have $\sigma_t^p \equiv 0$, since over the small time period dt, a δdt portion of firms get to change their prices and there is no stochastic change in aggregate price level p_t up to a first-order. With (E.3) and (B.5), the capitalist's consumption C_t follows

$$\frac{dC_t}{C_t} = \left(i_t + (\sigma + \sigma_t^q)^2 - \pi_t - \rho\right) dt + (\sigma_t + \sigma_t^q) dZ_t. \tag{E.22}$$

where we use the equilibrium condition in (B.6): $i_t^m = i_t + (\sigma + \sigma_t^q)^2$. Thus, we obtain

$$\begin{aligned} d\hat{Q}_t = d\hat{C}_t &= \left(i_t - \pi_t - \underbrace{\left(r_t^n - \frac{(\sigma + \sigma_t^q)^2}{2} + \frac{\sigma^2}{2} \right)}_{\equiv r_t^T} \right) dt + \sigma_t^q dZ_t \\ &= \left(i_t - \pi_t - r_t^T\right) dt + \sigma_t^q dZ_t. \end{aligned} \tag{E.23}$$

Since we have risk-premium levels $\text{rp}_t = (\sigma_t + \sigma_t^q)^2$ in the sticky price economy and $\text{rp}_t^n =$

σ^2 in the flexible price economy, we can express our risk-adjusted natural rate r_t^T as

$$r_t^T = r_t^n - \frac{1}{2}\left(\mathrm{rp}_t - \mathrm{rp}_t^n\right) = r_t^n - \frac{1}{2}\hat{rp}_t, \tag{E.24}$$

■

Proof of Proposition B.5. This result is a direct consequence of Blanchard and Kahn (1980) and Buiter (1984).

■

Proof of Proposition B.4. The proof strategy is similar to Proposition 1 in the main body. From (B.20), $\{\sigma_t^q\}$ process is written as

$$d\sigma_t^q = -\frac{\phi^2(\sigma_t^q)^2}{2(\sigma+\sigma_t^q)^3}dt - \phi\frac{\sigma_t^q}{\sigma+\sigma_t^q}dZ_t. \tag{E.25}$$

Using Ito's lemma on (E.25), we write the process for $(\sigma + \sigma_t^q)^2$, which is a martingale itself, as

$$\begin{aligned} d\left(\sigma+\sigma_t^q\right)^2 &= 2\left(\sigma+\sigma_t^q\right)d\sigma_t^q + \left(d\sigma_t^q\right)^2 \\ &= 2\left(\sigma+\sigma_t^q\right)\left(-\frac{\phi^2(\sigma_t^q)^2}{2\left(\sigma+\sigma_t^q\right)^3}dt - \phi\frac{\sigma_t^q}{\sigma+\sigma_t^q}dZ_t\right) + \phi^2\frac{(\sigma_t^q)^2}{(\sigma+\sigma_t^q)^2}dt \\ &= -2\phi\left(\sigma_t^q\right)dZ_t. \end{aligned} \tag{E.26}$$

Therefore, we would have $\mathbb{E}_0((\sigma+\sigma_t^q)^2) = (\sigma+\sigma_0^q)^2$ where $\mathbb{E}_0$ is an expectation operator with respect to the $t = 0$ filtration. By Doob's martingale convergence theorem (as $(\sigma + \sigma_t^q)^2 \geq 0, \forall t$), we know $\sigma_t^q \overset{a.s}{\to} \sigma_\infty^q = \sigma^{q,n} = 0$ since:

$$\underbrace{d\sigma_t^q}_{\overset{a.s}{\to}0} = -\underbrace{\frac{\phi^2\left(\sigma_t^q\right)^2}{2\left(\sigma+\sigma_t^q\right)^3}}_{\overset{a.s}{\to}0}dt - \phi\underbrace{\frac{\sigma_t^q}{\sigma+\sigma_t^q}}_{\overset{a.s}{\to}0}dZ_t. \tag{E.27}$$

Thus, (E.27) proves $\sigma_t^q \overset{a.s}{\to} \sigma_\infty^q = 0$. From (B.19) $\sigma_t^q \overset{a.s}{\to} \sigma_\infty^q = 0$ leads to $\hat{Q}_t \overset{a.s}{\to} 0$ and $\pi_t \overset{a.s}{\to} 0$. Finally, we should have $\mathbb{E}(\max_t(\sigma_t^q)^2) = \infty$, since otherwise, the uniform integrability implies $\mathbb{E}((\sigma+\sigma_\infty^q)^2) = (\sigma+\sigma_0^q)^2$, which is contradiction to our earlier result $\sigma_t^q \overset{a.s}{\to} \sigma^{q,n} = 0$ since $\sigma_\infty^q = 0$ and $\sigma_0^q > \sigma^{q,n} = 0$ by assumption in Proposition B.4.

■

Proof of Lemma B.2. From $C_t = \rho A_t Q_t$, we obtain $\hat{C}_t = \hat{Q}_t$. We start from the flexible price economy's good market equilibrium condition, where we use equation (E.2). Here $\frac{w_t^n}{p_t^n}$ is the real wage level in the flexible price economy. The good market equilibrium condition can be written as

$$A_t \left(\frac{w_t^n}{p_t^n}\right)^{\frac{1}{\chi}} \frac{1}{A_t^{\frac{1}{\chi}}} = \rho A_t Q_t^n + \left(\frac{w_t^n}{p_t^n}\right)^{1+\frac{1}{\chi}} \frac{1}{A_t^{\frac{1}{\chi}}}. \tag{E.28}$$

We subtract equation (E.28) from the same good market condition in the sticky price economy to obtain

$$A_t \left(\left(\frac{w_t}{p_t}\right)^{\frac{1}{\chi}} - \left(\frac{w_t^n}{p_t^n}\right)^{\frac{1}{\chi}}\right) \frac{1}{A_t^{\frac{1}{\chi}}} = (C_t - C_t^n) + \left(\left(\frac{w_t}{p_t}\right)^{1+\frac{1}{\chi}} - \left(\frac{w_t^n}{p_t^n}\right)^{1+\frac{1}{\chi}}\right) \frac{1}{A_t^{\frac{1}{\chi}}}, \tag{E.29}$$

where we divide both sides of equation (E.29) by $y_t^n \equiv A_t^{1-\frac{1}{\chi}} (\frac{w_t^n}{p_t^n})^{\frac{1}{\chi}}$ and obtain

$$\underbrace{\frac{\left(\frac{w_t}{p_t}\right)^{\frac{1}{\chi}} - \left(\frac{w_t^n}{p_t^n}\right)^{\frac{1}{\chi}}}{\left(\frac{w_t^n}{p_t^n}\right)^{\frac{1}{\chi}}}}_{=\frac{1}{\chi}\widehat{\frac{w_t}{p_t}}} = \underbrace{\frac{C_t^n}{A_t^{1-\frac{1}{\chi}} \left(\frac{w_t^n}{p_t^n}\right)^{\frac{1}{\chi}}}}_{=1-\frac{(\epsilon-1)(1-\alpha)}{\epsilon}} \hat{C}_t + \underbrace{\frac{\left(\frac{w_t}{p_t}\right)^{1+\frac{1}{\chi}} - \left(\frac{w_t^n}{p_t^n}\right)^{1+\frac{1}{\chi}}}{A_t \left(\frac{w_t^n}{p_t^n}\right)^{\frac{1}{\chi}}}}_{=\frac{(\epsilon-1)(1-\alpha)}{\epsilon}\left(1+\frac{1}{\chi}\right)\widehat{\frac{w_t}{p_t}}}, \tag{E.30}$$

which can be written as:

$$\frac{1}{\chi}\frac{\widehat{w_t}}{p_t} = \left(1 - \frac{(\epsilon-1)(1-\alpha)}{\epsilon}\right)\hat{C}_t + \frac{(\epsilon-1)(1-\alpha)}{\epsilon} \underbrace{\left(1+\frac{1}{\chi}\right)\frac{\widehat{w_t}}{p_t}}_{=\hat{C^w}(t)}. \tag{E.31}$$

Equation (E.31) with $\hat{C}_t = \hat{Q}_t$ leads to

$$\hat{Q}_t = \underbrace{\left(\chi^{-1} - \frac{\frac{(\epsilon-1)(1-\alpha)}{\epsilon}}{1 - \frac{(\epsilon-1)(1-\alpha)}{\epsilon}}\right)}_{>0} \frac{\widehat{w_t}}{p_t} = \underbrace{\frac{1}{1+\chi^{-1}} \left(\chi^{-1} - \frac{\frac{(\epsilon-1)(1-\alpha)}{\epsilon}}{1 - \frac{(\epsilon-1)(1-\alpha)}{\epsilon}}\right)}_{>0} \widehat{C_{W,t}}.$$

We observe that Assumption B.1 guarantees that gaps of asset price, consumption of capitalists and workers, employment, and real wage all co-move with positive correlations.

Now we can use $\hat{Q}_t$ and $\hat{C}_t$ interchangeably, and if one gap variable becomes 0, then all other gap variables become also stabilized to 0, up to a first order.

■

Proof of Proposition B.3. Firms change their prices with instantaneous probability δdt à la Calvo (1983). If there is price dispersion Δ_t, as defined in (B.2), across intermediate goods firms, then labor market equilibrium condition can be written as

$$N_{W,t} = \int_0^1 n_t(i)di = \left(\frac{y_t}{A_t\left(N_{W,t}\right)^{\alpha}}\right)^{\frac{1}{1-\alpha}} \underbrace{\int_0^1 \left(\frac{p_t(i)}{p_t}\right)^{-\frac{\epsilon}{1-\alpha}} di}_{\equiv \Delta_t^{\frac{1}{1-\alpha}}}, \tag{E.32}$$

where

$$y(t) = \frac{A_t N_{W,t}}{\Delta_t} = C_t + C_{W,t}. \tag{E.33}$$

We know that the good market equilibrium condition in (B.7) can be written as

$$\rho A_t Q_t + A_t \left(\frac{w_t}{p_t A_t}\right)^{1+\frac{1}{\chi}} = A_t \left(\frac{w_t}{p_t A_t}\right)^{\frac{1}{\chi}} \frac{1}{\Delta_t}. \tag{E.34}$$

Since a price process does not affect the resource allocation in the flexible price economy, we can regard $\hat{x}_t$ to be the $\log$-deviation of x_t from the flexible price economy *where the price is constant*. From the price aggregator, we obtain up to a first order

$$\hat{p}_t = \int_0^1 \widehat{p_t(i)} di. \tag{E.35}$$

To study price dispersion Δ_t up to a first-order, we illustrate Woodford (2003)'s treatment of Δ_t up to a second-order. From

$$\begin{aligned} \frac{1}{1-\alpha}\hat{\Delta}_t &= \ln \int_0^1 \left(1 - \frac{\epsilon}{1-\alpha}\left(\cancel{\widehat{p_t(i)} - \hat{p}_t}\right) + \frac{1}{2}\left(\frac{\epsilon}{1-\alpha}\right)^2 \left(\widehat{p_t(i)} - \hat{p}_t\right)^2\right) di + \text{h.o.t.} \\ &= \frac{1}{2}\left(\frac{\epsilon}{1-\alpha}\right)^2 Var_i\left(\widehat{p_t(i)}\right) + \text{h.o.t}, \end{aligned} \tag{E.36}$$

where h.o.t stands for higher-order terms, we observe that $\Delta_t \simeq 1$ up to a first-order because Δ_t is in nature the second order as (E.36) suggests. Pricing à la Calvo (1983) is standard, except that our model is in continuous time. For dt period from t to $t+dt$, individual firm i change the price with δdt probability. From time-0's perspective, the probability that firm resets its price for the first time at time t is

$$\delta \mathrm{e}^{-\delta t} dt = \underbrace{\delta dt}_{\text{Change now}} \cdot \underbrace{\mathrm{e}^{-\delta t}}_{\text{No change until t}} . \tag{E.37}$$

At time t, a price-changing firm i chooses $p_t(i)$ to solve

$$\begin{aligned} &\max_{p_t(i)} \frac{1}{\xi_t^N p_t} \mathbb{E}_t \int_t^\infty \mathrm{e}^{-\delta(s-t)} \xi_s^N p_s \left(\frac{p_t(i)}{p_s} y_{s|t}(i) - \frac{1}{p_s} C(y_{s|t}(i)) \right) ds, \text{ with } y_{s|t}(i) = \left(\frac{p_t(i)}{p_s} \right)^{-\epsilon} y_s \\ &= \frac{1}{\xi_t^N p_t} \mathbb{E}_t \int_t^\infty \mathrm{e}^{-\delta(s-t)} \xi_s^N p_s \left(\left(\frac{p_t(i)}{p_s} \right)^{1-\epsilon} y_s - \frac{1}{p_s} C\left(\left(\frac{p_t(i)}{p_s} \right)^{-\epsilon} y_s \right) \right) ds, \end{aligned} \tag{E.38}$$

where $C(\cdot)$ is defined as an individual firm's nominal production cost as a function of its output produced, which is to be written explicitly. Let $MC_{s|t}$ and $\varphi_{s|t}$ be the nominal and real marginal cost at time s conditional on the last price resetting at prior time t. Using the nominal pricing kernel ξ_s^N formula in (B.4), we obtain

$$\frac{\xi_s^N p_s}{\xi_t^N p_t} = \mathrm{e}^{-\rho(s-t)} \frac{C_t}{C_s}. \tag{E.39}$$

By plugging (E.39) into (E.38) and solving (E.38), the optimal adjusted price p_t^*[28] is given as

$$p_t^* = \frac{\mathbb{E}_t \displaystyle\int_t^\infty \mathrm{e}^{-(\delta+\rho)(s-t)} \frac{y_s}{C_s} \frac{\varphi_{s|t}}{\bar{\varphi}} p_s^\epsilon ds}{\mathbb{E}_t \displaystyle\int_t^\infty \mathrm{e}^{-(\delta+\rho)(s-t)} \frac{y_s}{C_s} p_s^{\epsilon-1} ds}, \tag{E.40}$$

where $\varphi_{s|t}$ appears, and $\bar{\varphi}$ is its level in the flexible-price equilibrium, which is $\frac{\epsilon-1}{\epsilon}$. If we log-linearize (E.40) around the flexible price equilibrium with constant price as in (E.35), we can log-linearize $\widehat{p_t^*}$ expressed as

$$\widehat{p_t^*} = (\delta+\rho) \mathbb{E}_t \int_t^\infty \mathrm{e}^{-(\delta+\rho)(s-t)} \left(\hat{\varphi}_{s|t} + \hat{p}_s \right) ds. \tag{E.41}$$

We know the conditional real production cost and the conditional real marginal cost can be written as

$$\frac{1}{p_s} C(y_{s|t}) = \frac{w_s}{p_s} \left(\frac{y_{s|t}}{A_s (N_{W,s})^\alpha} \right)^{\frac{1}{1-\alpha}}, \tag{E.42}$$

[28] We use the property that every price-setting firm at any time t chooses the same price, so we drop the firm index i in $p_t^*(i)$ and use p_t^*.

and

$$\varphi_{s|t} \equiv \frac{1}{p_s} C'(y_{s|t}) = \frac{w_s}{p_s} \left(\frac{y_{s|t}}{A_s (N_{W,s})^{\alpha}} \right)^{\frac{\alpha}{1-\alpha}} \frac{1}{A_s (N_{W,s})^{\alpha}}. \tag{E.43}$$

From equation (E.43)), we obtain the conditional real marginal cost gap at time s conditional on price resetting at time t, which is given by

$$\hat{\varphi}_{s|t} = \underbrace{\frac{\hat{w}_s}{p_s}}_{\equiv \hat{\varphi}_s} - \frac{\alpha \epsilon}{1-\alpha} \left(\widehat{p_t^*} - \hat{p}_s \right) = \hat{\varphi}_s - \frac{\alpha \epsilon}{1-\alpha} \left(\widehat{p_t^*} - \hat{p}_s \right). \tag{E.44}$$

where $\hat{\varphi}_s$ is defined as the aggregate marginal cost index: as production is linear in aggregate level, $\hat{\varphi}_s$ should be equal to the real wage gap. Using (E.35), we then characterize the change in aggregate price gap $\hat{p}_t$ as

$$\begin{aligned} d\hat{p}_t &= \delta dt \left(\widehat{p_t^*} - \hat{p}_t \right) \\ &= \delta dt (\delta + \rho) \mathbb{E}_t \int_t^{\infty} \mathrm{e}^{-(\delta+\rho)(s-t)} \left(\Theta \hat{\varphi}_s + \hat{p}_s - \hat{p}_t \right) ds, \text{ where } \Theta \equiv \frac{1-\alpha}{1-\alpha+\alpha\epsilon}. \end{aligned} \tag{E.45}$$

As we log-linearize our economy around the flexible price equilibrium with constant price (i.e., $\pi_t = \sigma_t^p = 0$), $\hat{p}_t$ changes with a rate of current π_t,[29] we have

$$\pi_t = \frac{d\hat{p}_t}{dt} = \delta(\delta+\rho) \mathbb{E}_t \int_t^{\infty} \mathrm{e}^{-(\delta+\rho)(s-t)} \left(\Theta \hat{\varphi}_s + \hat{p}_s - \hat{p}_t \right) ds. \tag{E.46}$$

Now that we have (E.46) for the instantaneous inflation π_t. we manipulate (E.46) as:

$$\begin{aligned} \pi_t + \delta \hat{p}_t &= \delta(\delta+\rho) \mathbb{E}_t \int_t^{\infty} \mathrm{e}^{-(\delta+\rho)(s-t)} (\Theta \hat{\varphi}_s + \hat{p}_s) ds \\ &= \delta(\delta+\rho) \mathrm{e}^{(\delta+\rho)t} \mathbb{E}_t \int_t^{\infty} \mathrm{e}^{-(\delta+\rho)s} \left(\Theta \hat{\varphi}_s + \hat{p}_s \right) ds \\ &= \delta(\delta+\rho)(\Theta \hat{\varphi}_t + \hat{p}_t) dt + \delta(\delta+\rho) \mathrm{e}^{(\delta+\rho)t} \mathbb{E}_t \int_{t+dt}^{\infty} \mathrm{e}^{-(\delta+\rho)s} \left(\Theta \hat{\varphi}_s + \hat{p}_s \right) ds, \end{aligned} \tag{E.47}$$

where we can rewrite the first line of equation (E.47) at time $t + dt$ instead of t as

[29] In the case of positive inflation targets, see e.g., Coibion et al. (2012).

$$\pi_{t+dt} + \delta\hat{p}_{t+dt} = \delta(\delta+\rho)\mathrm{e}^{(\delta+\rho)(t+dt)}\mathbb{E}_{t+dt}\int_{t+dt}^{\infty}\mathrm{e}^{-(\delta+\rho)s}\left(\Theta\hat{\varphi}_s+\hat{p}_s\right)ds$$
$$= \delta(\delta+\rho)\mathrm{e}^{(\delta+\rho)t}\left(1+(\delta+\rho)dt\right)\mathbb{E}_{t+dt}\int_{t+dt}^{\infty}\mathrm{e}^{-(\delta+\rho)s}\left(\Theta\hat{\varphi}_s+\hat{p}_s\right)ds. \tag{E.48}$$

Due to the *martingale representation theorem* (see e.g., Oksendal (1995)), there exists a measurable H_t such that

$$\mathbb{E}_{t+dt}\int_{t+dt}^{\infty}\mathrm{e}^{-(\delta+\rho)s}(\Theta\hat{\varphi}_s+\hat{p}_s)ds = \mathbb{E}_t\int_{t+dt}^{\infty}\mathrm{e}^{-(\delta+\rho)s}(\Theta\hat{\varphi}_s+\hat{p}_s)ds + H_t dZ_t, \tag{E.49}$$

holds. We plug (E.49) into equation (E.48) to obtain[30]

$$\pi_{t+dt} + \delta\hat{p}_{t+dt} = \delta(\delta+\rho)\Bigg(\mathrm{e}^{(\delta+\rho)t}\mathbb{E}_t\int_{t+dt}^{\infty}\mathrm{e}^{-(\delta+\rho)s}(\Theta\hat{\varphi}_s+\hat{p}_s)ds + \mathrm{e}^{(\delta+\rho)t}H_t dZ_t$$
$$+ \mathrm{e}^{(\delta+\rho)t}(\delta+\rho)dt\cdot\mathbb{E}_t\int_{t+dt}^{\infty}\mathrm{e}^{-(\delta+\rho)s}(\Theta\hat{\varphi}_s+\hat{p}_s)ds\Bigg). \tag{E.50}$$

We subtract (E.47) from (E.50) to obtain

$$d\pi_t + \cancel{\delta}\pi_t dt$$
$$= \delta(\delta+\rho)\left(\mathrm{e}^{(\delta+\rho)t}(\delta+\rho)dt\cdot\mathbb{E}_t\int_{t+dt}^{\infty}\mathrm{e}^{-(\delta+\rho)s}(\Theta\hat{\varphi}_s+\hat{p}_s)ds + \mathrm{e}^{(\delta+\rho)t}H_t dZ_t - (\Theta\hat{\varphi}_t+\hat{p}_t)dt\right)$$
$$= \underbrace{\delta(\delta+\rho)\mathrm{e}^{(\delta+\rho)t}H_t}_{\equiv\sigma_{\pi,t}} dZ_t - \delta(\delta+\rho)\Theta\hat{\varphi}_t dt$$
$$+ \underbrace{\delta(\delta+\rho)\left((\delta+\rho)dt\cdot\mathbb{E}_t\int_{t+dt}^{\infty}\mathrm{e}^{-(\delta+\rho)(s-t)}(\Theta\hat{\varphi}_s+\hat{p}_s-\hat{p}_t)ds\right)}_{=(\cancel{\delta}+\rho)\pi_t dt}, \tag{E.51}$$

where we use

$$(\delta+\rho)dt\cdot\mathbb{E}_t\int_{t+dt}^{\infty}\mathrm{e}^{-(\delta+\rho)(s-t)}(\Theta\hat{\varphi}_s+\hat{p}_s-\hat{p}_t)ds = (\delta+\rho)dt\cdot\mathbb{E}_t\int_{t}^{\infty}\mathrm{e}^{-(\delta+\rho)(s-t)}(\Theta\hat{\varphi}_s+\hat{p}_s-\hat{p}_t)ds, \tag{E.52}$$

which holds from the property $(dt)^2 = 0$. Note that in (E.51), we define $\sigma_{\pi,t}$ as an instanta-

[30] We use the property that $dt\cdot dZ_t = 0$.

neous volatility of the inflation process. Finally from equation (E.51) we get the continuous time version of New Keynesian Phillips curve (NKPC), written as[31]

$$d\pi_t = \rho\pi_t dt - \delta(\delta+\rho)\Theta\hat{\varphi}_t dt + \sigma_{\pi,t} dZ_t. \tag{E.53}$$

Due to the linear aggregate production function up to a first-order, we obtain:[32]

$$\hat{\varphi}_t = \frac{\widehat{w_t}}{p_t} = \left(\chi^{-1} - \frac{\frac{(\epsilon-1)(1-\alpha)}{\epsilon}}{1-\frac{(\epsilon-1)(1-\alpha)}{\epsilon}}\right)^{-1} \hat{Q}_t \equiv \frac{\kappa}{\delta(\delta+\rho)\Theta}\hat{Q}_t. \tag{E.54}$$

Finally plugging equation (E.54) into equation (E.53), we represent New-Keynesian Phillips curve in terms of asset price gap $\hat{Q}_t$ in the following way:

$$d\pi_t = \left(\rho\pi_t - \kappa\hat{Q}_t\right) dt + \sigma_{\pi,t} dZ_t, \text{ and } \mathbb{E}_t d\pi_t = \left(\rho\pi_t - \kappa\hat{Q}_t\right) dt, \tag{E.55}$$

which proves the proposition B.3.[33] We know $\kappa > 0$ due to Assumption B.1. ■

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[31] Our continuous-time version of the Phillips curve in (E.51) is of the same form as in Werning (2012) and Cochrane (2017) after taking expectation on both sides.

[32] We use Lemma 2's log-linearization result to represent the real aggregate marginal cost gap $\frac{\hat{w}_t}{p_t}$ as a function of capitalists' consumption gap $\hat{C}_t = \hat{Q}_t$.

[33] Since $\hat{y}_t = \zeta\hat{Q}_t$, Phillips curve can be represented in terms of output gap $\hat{y}_t$ as in Proposition B.3.

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